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David Middleton
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The following table shows the results of the regression analysis for the dependent variable "Number of children in the household" (N = 1,000). The independent variables are "Age of the head of household" and "Gender of the head of household". The table includes the coefficient estimates, standard errors, t-statistics, and p-values for each variable.

Variable	Coefficient	Standard Error	t-statistic	p-value
Age of the head of household	0.001	0.001	1.2	0.23
Gender of the head of household	0.002	0.001	1.5	0.13
Constant	1.5	0.5	3.0	0.00
Adjusted R-squared	0.00			

THE UNIVERSITY OF CHICAGO

PREFACE

This research was conducted under NUSC Project No. A70272, Subproject No. RR0000-N01, Selected Statistical Problems in Acoustic Signal Processing, Principal Investigator Dr. Albert H. Nuttall (Code 304). This technical report was prepared with funds provided by the NUSC In-House Independent Research and Independent Exploratory Development Program, sponsored by the Office of the Chief of Naval Research. Also, this research is based in part on work by Dr. David Middleton, supported originally by Code 10 under ONR Contract N00014-84-C-0417 with Code 1111.

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ATD for Surface Antisubmarine Warfare Directorate

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13. ABSTRACT (Maximum 200 words) Because of the critical rôle of non-Gaussian noise processes in modern signal processing, which usually involves nonlinear operations, it is important to examine the effects of the latter on such noise and the extension to added signal inputs. Here, only non-Gaussian (specifically Class A) noise inputs, with an additive Gaussian component, are considered. The "classical" problems of zero-memory nonlinear (ZMNL) devices serve to illustrate the approach and to provide a variety of useful output statistical quantities, e.g., mean or "dc" values, mean intensities, covariances, and their associated spectra. Here, Gaussian and non-Gaussian noise <u>fields</u> are introduced, and their respective temporal and spatial outputs are described and numerically evaluated for representative parameters of the noise and the ZMNL devices. Similar				
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13. ABSTRACT (continued)

statistics for carriers which are phase or frequency modulated by Class A (and Gaussian) noise are also presented, numerically evaluated, and illustrated in the figures. A series of appendices and programs provide the technical support for the numerical analysis.

14. SUBJECT TERMS (continued)

zero memory nonlinearity	with law rectification
frequency modulation	phase modulation

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LIST OF PRINCIPAL SYMBOLS

A	overlap index (A_A) of Class A noise
$\alpha(R,t)$	input (noise) field
B_v	output covariance function, cf. (2.7a)
b	dimensionless factor, (2.44) et seq.
D_P, D_F	phase- and frequency-modulation scaling factors
δ	(Dirac) delta function
F_1, F_2	characteristic functions
FM	frequency modulation
$f(i\xi)$	Fourier transform of rectifier transfer function
g	ZMNL input-output transfer function
Γ'	$\sigma_G^2/\Omega_{2A} =$ "Gaussian" factor
$H_1^{(v)}$	scaling parameter for v -th law detector outputs, (2.18a, b, c)
J_0	Bessel function of first kind
K, K_L, K_G	covariances
k_L, k_G	normalized covariances
ξ, ξ_1, ξ_2	characteristic function variables
$M_Y, \hat{M}_Y, \hat{M}'_Y$	second-moment functions, (2.7), (2.8), (2.11)
$\mu_F, \hat{\mu}_F$	frequency-modulation indexes
$\mu_P, \hat{\mu}_P$	phase-modulation indexes
v	power law of half-wave rectifier
\hat{v}, \hat{v}	normalized wavenumber, cf. (2.33a)
$\Omega_{G,A}$	covariance functions (angle-modulation), section 2.2-6

Ω_{2A}	intensity of Class A noise
Ω_O	spectral parameter in FM, PM, (2.42c)
PM	phase modulation
$\Delta R, \hat{\Delta R}$	spatial correlation distances
\hat{R}	array operator
Re z	real part of z
ρ	"overlap" correlation function, (2.3)
σ_{m+n}^2	noise variance
σ_G^2	intensity of Gauss noise component
T	observation time
T_s	waveform duration of elementary noise source
τ	$t_2 - t_1$, delay (correlation) time
\hat{f}, \hat{f}'	normalized delay parameters
$w(\hat{\omega})$	normalized intensity spectrum, (2.52)
w_2, \hat{w}_2	(wavenumber) intensity spectra
$w_{()}, \hat{w}_{()}$	(frequency) intensity spectra
$x, x(t)$	array outputs = inputs to ZMNL devices
$y, y(t)$	outputs of ZMNL devices
Y_a	correlation parameter, cf. (2.7b)
ZMNL	zero-memory nonlinear

LIST OF NORMALIZING AND NORMALIZED PARAMETERS

1. $\tau = \beta \tau$; $\beta = 1/\bar{T}_S$; $\tau = t_2 - t_1$: correlation delay
 \bar{T}_S = mean duration of typical interfering signal
2. $\rho = 1 - |\hat{r}|$ for $|\hat{r}| < 1$, zero otherwise; (2.3): "overlap" correlation function
3. $\hat{\Delta R} = \Delta R / \Delta_L$ (≥ 0); $\Delta R = |R_2 - R_1|$, correlation distance
4. $\hat{\omega} = \omega / \beta = \omega \bar{T}_S$; $\omega = 2\pi f$: normalized (angular) frequency
5. $\hat{\Delta \omega}_L = \Delta \omega_L / \beta$; normalized (frequency) spectrum bandwidth, for Class A noise model
6. $\hat{\Delta \omega}_G = \Delta \omega_G / \beta$; normalized (frequency) spectrum bandwidth, for Gauss noise component
7. Δ_L = rms spread of spatial covariance of non-Gaussian noise field component
8. Δ_G = rms spread of spatial covariance of Gaussian noise field component
9. $\hat{k} = k \Delta_L$: normalized wavenumber
10. $\Delta \omega_N$ = spectrum bandwidth of modulating (RC-Gauss) noise; cf. (2.43) and [1; section 14.1-3]; cf. (11) ff.
11. $\Delta \omega_N \rightarrow \Delta \omega_{N(L)}$ = bandwidth of non-Gaussian component of modulating noise in PM and FM;
 $\Delta \omega_N \rightarrow \Delta \omega_{N(G)}$ = bandwidth of Gaussian component of modulating noise in PM and FM
12. $\zeta = \tau \Delta \omega_N$, correlation variable, cf. (2.43) and (A.6-9) and figures 3.9a through 3.10b; see (A.6-9) for ζ_c
13. $\hat{r}' = \hat{r} - \beta(\Delta R / c_0)$, with $\hat{r} = \tau - \Delta R / c_0$, cf. (2.3a)
14. $\hat{\mu}_F = \mu_F(1 + \Gamma')^{\frac{1}{2}}$: normalized FM index; (2.53)
15. $\hat{\mu}_P = \mu_P(1 + \Gamma')^{\frac{1}{2}}$: normalized PM index; (2.53)
16. $\hat{\omega} = (\omega - \omega_0) / \Delta \omega_N$ = normalized displaced angular frequency, cf. (2.49)

SPECTRA AND COVARIANCES FOR "CLASSICAL" NONLINEAR SIGNAL
PROCESSING PROBLEMS INVOLVING CLASS A NON-GAUSSIAN NOISE

PART I. ANALYTIC RESULTS AND NUMERICAL EXAMPLES

1. INTRODUCTION

Non-Gaussian noise fields play a critical rôle in modern signal processing because of the frequently dominant effects of such noise and interference in a wide variety of applications. Communication theory generally, and specifically telecommunications, electromagnetic and acoustic scattering, man-made and natural ambient noise, optics, and underwater acoustics, are common areas of interest in this respect. In the present report we are concerned primarily with underwater acoustic noise phenomena, but the models and results are canonical, that is, they take forms invariant to the particular physical application in question.

Specifically, we are concerned with various second-order statistics of non-Gaussian noise processes and fields after they have been subjected to different types of nonlinear operations, such as rectification and modulation. A generic problem here is the passage of non-Gaussian noise through a zero-memory nonlinear (ZMNL) device. The desired output statistics are typically the mean (dc), mean intensity (power), the covariance or correlation function, and the associated spectra. These last include wavenumber spectra in the case of noise fields, as well as the more general frequency-wavenumber spectra obtained by joint temporal and spatial Fourier transformations. Typical "classical" problems include: (i) rectification, (ii) determination of output spectra and covariances, (iii) calculation of (output) signal-to-noise ratios, (iv) modulation, (v) demodulation, and (vi) special systems, as for example, the spectrum analyzer. These and other problems involving ZMNL devices are described in detail in [1; chapters 5 and 12 - 17]. What is new here is the

use of the approximate second-order probability density functions and characteristic functions in the above applications when the noise processes are non-Gaussian.

A full treatment is given in a current study by Middleton, [2], which is an expanded version of his recent paper [3], which employs some of the results of the present report, namely, the calculated covariances and spectra. Here, we are content to summarize the pertinent analytic results, the corresponding examples of calculated covariances and spectra, and the various computational procedures associated with their evaluation. The details of the derivations are provided in [2] and [3]. Included here, also, is a selection of illustrations of the analytic results.

2. ANALYTICAL RESULTS: A SUMMARY

In the present study, we address three classical problems where the goals are the calculation of the covariance and associated intensity spectrum. Specifically, we consider:

Problem I. The half-wave v -th law rectification of Class A noise fields and processes;

Problem II. Phase modulation of a carrier by a Class A noise process; and

Problem III. Frequency modulation of a carrier by a Class A noise process.

Class A noise, as noted in section 3 of [2], [3], is a canonical form of interference characterized by a coherent structure vis-à-vis the (linear) front-end stages of a typical receiver: negligible transients are produced at the output of these stages. Class B noise, on the other hand, is incoherent and highly impulsive, such that the front-end stages of the receiver generate an output which consists solely of (overlapping) transients. Here, the Class A models are tractable in the required second-order distribution and characteristic functions, whereas the Class B models are not and must

consequently be appropriately approximated in second-order; see [4] and [5] for additional information. In the present report, we shall consider examples of Class A noise only.

2.1 THE SECOND-ORDER CLASS A CHARACTERISTIC FUNCTION

In applications [1] - [3], the second-order characteristic function, $F_2(i\xi_1, i\xi_2)$, plays a key rôle: from it, we may obtain the aforementioned statistics of the outputs of ZMNL devices, spectra of angle-modulated carriers, and other usually second-order statistics of various nonlinear operations arising in a variety of communication and measurement operations.

(See [2], [3] for further discussion.)

Here, we specifically use the approximate Class A noise characteristic function, F_2 , including an additive Gaussian component, given by

$$F_2(i\xi_1, i\xi_2)_{A+G} = \exp[-A(2-\rho)] \sum_{m_1, m_2=0}^{\infty} \frac{[A(1-\rho)]^{m_1+m_2}}{m_1! m_2!} \times \sum_{n=0}^{\infty} \frac{(A\rho)^n}{n!} \exp\left[-\frac{1}{2} Q_{m_1+n, m_2+n}^{(2)}(\xi_1, \xi_2)\right], \quad (2.1)$$

where $A (=A_A)$ is the "overlap" index, and where

$$Q_{m_1+n, m_2+n}^{(2)}(\xi_1, \xi_2) \equiv \xi_1^2 \sigma_{m_1+n}^2 + \xi_2^2 \sigma_{m_2+n}^2 + 2\xi_1 \xi_2 K_{L+G}^{(n)}, \quad (2.2a)$$

$$\sigma_{m+n}^2 \equiv \left(\frac{m+n}{A} + \Gamma'_A\right) \Omega_{2A}; \quad \Omega_{2A} \equiv \frac{1}{2} A \langle B_0^2 \rangle = A \langle L^2 \rangle; \quad \Gamma'_A = \sigma_G^2 / \Omega_{2A}; \quad (2.2b)$$

$$K_{L+G}^{(n)} = \left(n k_L / A + k_G \Gamma'_A\right) \Omega_{2A}; \quad (2.2c)$$

and k_L, k_G are the normalized covariances of the non-Gauss and Gauss components, respectively. Thus, $|k_{L,G}| \leq 1$.

Here, ρ ($=\rho_A$) is the "overlap" correlation function

$$\rho(\tau') \equiv \begin{cases} 1 - \beta|\tau'| & \text{for } \beta|\tau'| \leq 1 \\ 0 & \text{for } \beta|\tau'| > 1 \end{cases}, \quad \beta = 1/\bar{T}_s, \quad (2.3)$$

in which \bar{T}_s is the mean duration of a typical noise-signal of intensity $\langle B_0^2 \rangle / 2 = \langle L^2 \rangle$. The time delay τ' is given by

$$\tau' = \tau - \frac{\Delta R}{c_0} \quad \text{or} \quad \tau' = \tau (= t_2 - t_1), \quad (2.3a)$$

respectively, for space-time fields and received temporal processes. The path delay $\Delta R/c_0$ ($= |R_2 - R_1|/c_0$) accounts for the time differential between propagation paths to the points at which processing occurs, cf. figure 2.1 ff, Case A. The quantities Ω_{2A} and σ_G^2 are, respectively, the intensity of the non-Gaussian and Gaussian components which constitute the general Class A model used here. (However, we note that the present Class A model belongs either to the strictly canonical Class A cases, where all interfering sources are equidistant from the observer, or more generally, to the much broader class of situations in which the effective source distribution is concentrated in an annulus whose inner-to-outer radii have a ratio $O(1/2)$ or less. The former is exactly represented by (2.1) to second-order, while the latter is approximately so represented, albeit a good approximation as long as the aforementioned source annulus is not too large. See [5; section V, C], for example. For an exact treatment, see also [6], in an important class of physical models. Finally, differentiation of F_2 , (2.1), in the usual way, gives us the (exact) covariance of the composite Class A and Gauss field, namely,

$$K_{A+G} = - \frac{\partial^2}{\partial \xi_1 \partial \xi_2} F_2 \Big|_{\xi_1 = \xi_2 = 0} = K_A + K_G, \quad (2.4)$$

which in normalized form is

$$k_{A+G}(\Delta R, \tau) = \frac{k_L + \Gamma' k_G}{1 + \Gamma'} . \quad (2.4a)$$

In practice, A is usually less than unity, say 0(0.1 - 0.3) typically, so that only a comparatively few terms in $A^{m_1+m_2+n}$ are needed for numerical evaluation of (2.1) and the statistical quantities derived from it, cf. section 2.2 ff. Note that when $\beta|\tau| \geq 1$, $\rho = 0$, and $\Gamma' = 0$, we get

$$\begin{aligned} F_{2-A} &= \left(e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} \exp\left(-\frac{1}{2}\xi_1^2 \sigma_m^2\right) \right) \left(e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \exp\left(-\frac{1}{2}\xi_2^2 \sigma_n^2\right) \right) \\ &= F_1(i\xi_1)_A F_1(i\xi_2)_A , \end{aligned} \quad (2.4b)$$

as expected: there is now no correlation between process samples. With a Gaussian component, these will be correlated, of course, unless $|\tau| \rightarrow \infty$, so that $k_G \rightarrow 0$, cf. (2.2c).

2.2 PROBLEM I: HALF-WAVE v -TH LAW RECTIFICATION (STATIONARY AND HOMOGENEOUS FIELDS)

Here we consider the problem of obtaining the second-order (second-moment) statistics, M_y , of a sampled noise field, $\alpha(R, t)$, after passage through a ZMNL device, g , when the noise is generally non-Gaussian. Various processing configurations are possible. We show two in figure 2.1, below. Analytically, we have, for stationary and homogeneous inputs [1; section 2.3-2]

$$\begin{aligned} M_y(\Delta R, \tau) &= \overline{g(x_1)g(x_2)} = \frac{1}{(2\pi)^2} \int_{C_1} \int_{C_2} f(i\xi_1) f(i\xi_2) \\ &\times F_2(i\xi_1, i\xi_2; \Delta R, \tau)_x d\xi_1 d\xi_2 = \overline{y_1 y_2} , \end{aligned} \quad (2.5)$$

where $\Delta R = R_2 - R_1$, $\tau = t_2 - t_1$, and $f(i\xi)$ is the Fourier transform of the ZMNL device with $y_1 = y(R_1, t_1)$, etc. In the present cases, we have specifically

$$f(i\xi) = \beta \Gamma(v+1)/(i\xi)^{v+1}, \quad v > -1, \quad (2.6)$$

for these half-wave v -th law rectifiers [1; (2.101a,b)].

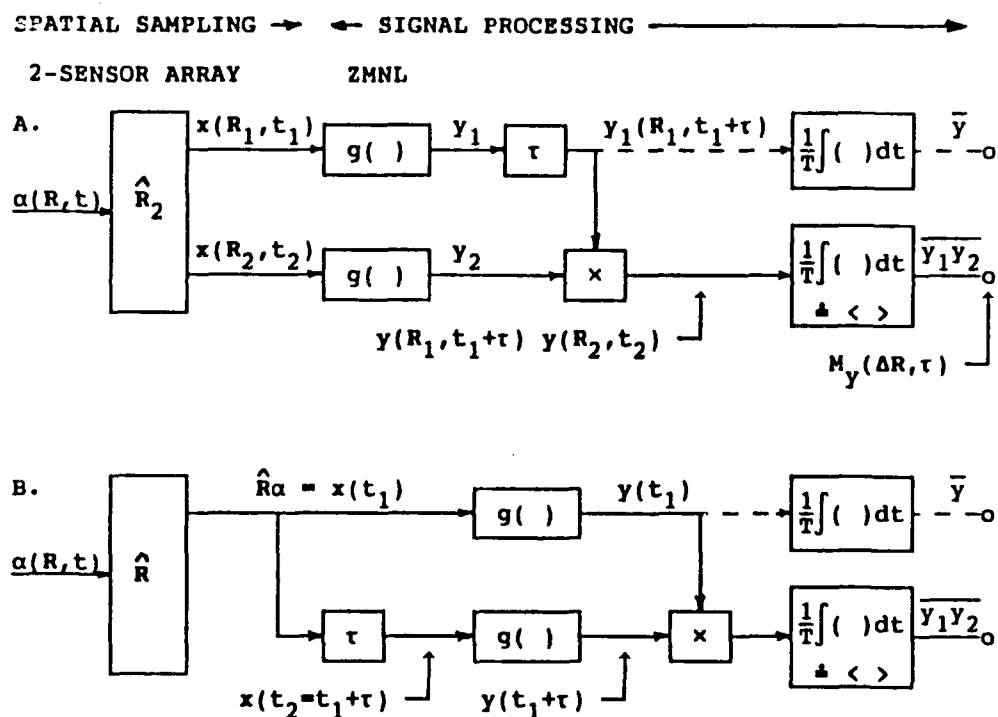


Figure 2.1 A. Two-point sensor array (\hat{R}_2) giving sampled field at two space-time points. B. A general array (\hat{R}) (preformed beam), converting the field $\alpha(R, t)$ into a single (time) process $x(t_1)$. Both are followed by ZMNL devices, delays, and averaging, as indicated schematically.

For the Class A non-Gaussian noise inputs of section 2.1 above, we find that the (normalized) second-moment M_Y for the resulting rectified field is now

$$M_Y(\hat{\Delta R}, \tau) = \exp[-A(2-\rho)] \sum_{m_1, m_2=0}^{\infty} \frac{[A(1-\rho)]^{m_1+m_2}}{m_1! m_2!} \sum_{n=0}^{\infty} \frac{(A\rho)^n}{n!} \\ \times \left(\frac{n+m_1}{A} + \Gamma' \right)^{\nu/2} \left(\frac{n+m_2}{A} + \Gamma' \right)^{\nu/2} B_\nu \Big|_{m_1, m_2, n} , \quad (2.7)$$

where we have further postulated the noise field to be isotropic, $\Delta R \rightarrow |\Delta R|$, and where specifically,

$$B_\nu \Big|_a \equiv B_\nu(Y|m_1, m_2, n) = \Gamma^2 \left(\frac{\nu+1}{2} \right) {}_2F_1 \left(-\frac{\nu}{2}, -\frac{\nu}{2}; \frac{1}{2}; Y_a^2 \right) \\ + 2Y_a \Gamma^2 \left(\frac{\nu}{2} + 1 \right) {}_2F_1 \left(\frac{1-\nu}{2}, \frac{1-\nu}{2}; \frac{3}{2}; Y_a^2 \right) , \quad (2.7a)$$

$$Y_a \equiv \frac{\frac{n}{A} k_L + \Gamma' k_G}{\left(\frac{m_1+n}{A} + \Gamma' \right)^{\frac{1}{2}} \left(\frac{m_2+n}{A} + \Gamma' \right)^{\frac{1}{2}}} ; \quad a = (m_1, m_2, n) , \quad |Y_a| \leq 1 . \quad (2.7b)$$

Specifically, also, we have the following normalized forms

$$\hat{M}_Y \equiv M_Y / \Omega_A^\nu 2^\nu / 4\pi ; \quad \hat{\tau}' \equiv \beta \tau' , \quad \beta = 1/\bar{T}_S , \quad \text{cf. (2.3) ,}$$

$$\hat{\Delta R} \equiv \Delta R / \Delta_L , \quad \Delta_L = \text{correlation distance,} \quad \Delta R = |R_2 - R_1| . \quad (2.8)$$

For numerical results, we select the following models for the space-time covariance functions of the isotropic and stationary non-Gaussian and Gaussian components of the input noise field:

$$k_L \equiv \exp \left(-\Delta R^2 / \Delta_L^2 - \frac{1}{2} (\Delta \omega_L \tau' / \beta)^2 \right) = \exp \left(-\hat{\Delta R}^2 - \frac{1}{2} (\hat{\Delta \omega}_L \hat{\tau}')^2 \right) ; \quad (2.9a)$$

$$k_G \equiv \exp \left(-\Delta R^2 / \Delta_G^2 - \frac{1}{2} (\Delta \omega_G \tau' / \beta)^2 \right) = \exp \left(-\hat{\Delta R}^2 (\Delta_L / \Delta_G)^2 - \frac{1}{2} (\hat{\Delta \omega}_G \hat{\tau}')^2 \right) ;$$

$$\hat{\Delta\omega}_L \equiv \Delta\omega_L/\beta, \quad \hat{\Delta\omega}_G \equiv \Delta\omega_G/\beta. \quad (2.9b)$$

Here, Δ_G is a correlation distance, and $\Delta\omega_L$, $\Delta\omega_G$ are angular frequency spreads associated with the respective non-Gaussian and Gaussian components of the input field. Note that if we define the correlation distance Δ_L as that where $k_L = 1/e$ ($\hat{r}' = 0$), then $\Delta_L = \Delta R_L$, etc.

For the special cases of v considered here, we also observe (from [1; (A.1-39)]) that B_v may be expressed in closed form:

$$B_0(Y) = \pi + 2 \arcsin(Y), \quad (2.10a)$$

$$B_1(Y) = Y \arcsin(Y) + \left(1 - Y^2\right)^{\frac{1}{2}} + \frac{\pi}{2}Y, \quad (2.10b)$$

$$B_2(Y) = \left(\frac{1}{2} + Y^2\right) \left(\frac{\pi}{2} + \arcsin(Y)\right) + \frac{3}{2}Y \left(1 - Y^2\right)^{\frac{1}{2}}. \quad (2.10c)$$

2.2-1 GAUSS PROCESSES ALONE ($A=0$)

When only a Gauss noise field is originally present, that is, $A = 0$, for example, $\Omega_{2A} = 0$, (2.7) reduces to the classical result [1; page 541, (13.4a)]:

$$\hat{M}'_y \equiv \hat{M}_y|_{A=0} = B_v|_{a=0}; \quad M_y = \frac{(2\psi)^v}{4\pi} \hat{M}_y; \quad Y_a \rightarrow Y_0 = k_G. \quad (2.11)$$

For comparison with the non-Gaussian cases ($A>0$), we choose to have equal input noise intensities. This means that

$$\psi_{A=0} = \sigma_G^2 + \Omega_{2A} = \Omega_{2A}(1 + \Gamma'),$$

so that

$$\hat{M}_y|_{A=0} = (1 + \Gamma')^v B_v|_{a=0}, \quad Y_0 = k_G, \quad (2.12)$$

and \hat{M}'_y is then to be compared with \hat{M}_y , $A > 0$. When Γ' is small, as is usually the case, we can often replace $(1 + \Gamma')^v$ by unity.

At this point, following figure 2.1, we distinguish two classes of operation: (A), where a pair of point sensors is used to sample the noise field and we wish to consider both the space and temporal correlations of the sampled field at the two points (R_1, t_1) , (R_2, t_2) ; and (B), where the space-time field is converted into a random process, $x(t)$, by the beamforming array (\hat{R}) , with an associated directionality embodied in the resultant beam (vide [7; sections IV B and VI A]).

2.2-2 CASE B, FIGURE 2.1

Let us consider the simpler case (Case B) of the time process first, cf. (B). For this, we set $\Delta R = 0$ formally in (2.7) et seq. above, since $x = \hat{R} \alpha(R, t)$ here and $\tau' = \tau = t_2 - t_1$, cf. (2.3a). See also [3; (3.2) et seq. and (3.11a)]. Then our ad hoc illustrative models of the process covariances k_L , k_G , are, from (2.9a,b), at once*

$$k_L = k_L(\tau) = \exp\left[-\frac{1}{2}(\Delta\omega_L \hat{r}/\beta)^2\right] = \exp\left[-\frac{1}{2}(\Delta\hat{\omega}_L \hat{r})^2\right], \quad (2.13a)$$

$$k_G = k_G(\tau) = \exp\left[-\frac{1}{2}(\Delta\omega_G \hat{r}/\beta)^2\right] = \exp\left[-\frac{1}{2}(\Delta\hat{\omega}_G \hat{r})^2\right]. \quad (2.13b)$$

Accordingly, (2.7) reduces to

$$\text{Case B: } M_Y(0, \hat{r}) \equiv M_Y(\hat{r})_B \approx (2.7), \text{ with } Y_a = (2.7b),$$

$$\text{and } (2.13a,b) \text{ and } \Delta\hat{R} = 0 \text{ therein.} \quad (2.14)$$

We note that when $|\hat{r}| \geq 1$, $\rho = 0$, and $\hat{M}_Y(0, |\hat{r}| \geq 1)$ reduces to a simpler relation [vis-à-vis (2.7)], viz.:

* A physically derived model of k_G and k_L may be made from [3; (3.11a)] with $L = \hat{R} L$, $\hat{R} = (2.9)$ etc., where L is typically given by [3; (3.3)], for example.

$$\hat{M}_Y(\hat{r})_B = e^{-2A} \sum_{m_1, m_2=0}^{\infty} \frac{A^{m_1+m_2}}{m_1! m_2!} \left(\frac{m_1}{A} + \Gamma' \right)^{v/2} \left(\frac{m_2}{A} + \Gamma' \right)^{v/2} B_v|_a, \quad (2.14a)$$

where (2.7b) becomes

$$Y_a = \frac{\Gamma' k_G}{\left(\frac{m_1}{A} + \Gamma' \right)^{1/2} \left(\frac{m_2}{A} + \Gamma' \right)^{1/2}}, \quad \rho = 0, \quad (2.14b)$$

in $B_v|_a$.

Special cases of interest are:

I. THE INTENSITY $E(Y^2)$: $\hat{r} = 0$, $\rho = 1$, $m_1 = m_2 = 0$, and (2.7), (2.14) reduce to

$$\overline{Y^2}|_{\text{norm}} = \hat{M}_Y(0)_B = \hat{M}_Y(0,0) = B_v|_{a=n} = e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma' \right)^v, \quad (2.15)$$

where now $Y_{a=n} = 1$, e.g., $k_L(0) = 1$ etc., and B_v is independent of n , for example, for $Y_a = 1$,

$$B_v|_{a=n} = \begin{cases} 2\pi & \text{for } v = 0 \\ \pi & \text{for } v = 1 \\ 3\pi/2 & \text{for } v = 2 \end{cases}, \text{ cf. (2.10)}. \quad (2.16a)$$

For general v , $Y_a = 1$, we have (from [1; (A.1-34)])

$$B_v|_{a=n} = 2\pi^{1/2} \Gamma(v+1/2), \quad v \geq 0. \quad (2.16b)$$

Thus, (2.15) becomes

$$\overline{Y^2}|_{\text{norm}} = \hat{M}_Y(0)_B = \hat{M}_Y(0,0) = 2\pi^{1/2} \Gamma(v+1/2) e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma' \right)^v. \quad (2.17)$$

The unnormalized form is, from (2.8),

$$\overline{y^2} = M_Y(0)_B = M_Y(0,0) = \frac{2^{v-1}}{\pi^{\frac{1}{2}}} \Omega_{2A}^v \Gamma(v+\frac{1}{2}) H_1^{(v)}(A, \Gamma') \quad (2.18)$$

with

$$H_1^{(v)}(A, \Gamma') \equiv e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma' \right)^v$$

$$= \begin{cases} 1 & \text{for } v = 0, \\ 1 + \Gamma' & \text{for } v = 1, \\ 1/A + (1+\Gamma')^2 & \text{for } v = 2. \end{cases} \quad (2.18a)$$

$$= \begin{cases} 1 & \text{for } v = 0, \\ 1 + \Gamma' & \text{for } v = 1, \\ 1/A + (1+\Gamma')^2 & \text{for } v = 2. \end{cases} \quad (2.18b)$$

$$= \begin{cases} 1 & \text{for } v = 0, \\ 1 + \Gamma' & \text{for } v = 1, \\ 1/A + (1+\Gamma')^2 & \text{for } v = 2. \end{cases} \quad (2.18c)$$

For other values of v (>0), we must evaluate $H_1^{(v)}$ numerically.

II. THE MEAN VALUE, \bar{y} ; $|\hat{t}| \rightarrow \infty$

Now $\rho = 0$, $n = 0$, $Y_a = 0$, and (2.7) reduces directly, upon use of (2.18), to

$$\begin{aligned} \overline{y^2} \Big|_{\text{norm}} &= \hat{M}_Y^{(\infty)}{}_B = \hat{M}_Y(0, \infty) = \Gamma^2 \left(\frac{v+1}{2} \right) \left\{ e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} \left(\frac{m}{A} + \Gamma' \right)^{v/2} \right\}^2 \\ &= \Gamma^2 \left(\frac{v+1}{2} \right) H_1^{(v/2)}(A, \Gamma')^2. \end{aligned}$$

(2.19)

The unnormalized form of (2.19) is, from (2.8),

$$\overline{y^2} = M_Y^{(\infty)}{}_B = M_Y(0, \infty) = \frac{2^v \Omega_{2A}^v}{4\pi} \Gamma^2 \left(\frac{v+1}{2} \right) H_1^{(v/2)}(A, \Gamma')^2, \quad (2.20)$$

and for v even, we find, from (2.18a,b,c)

$$H_1^{(0)} = 1, \quad H_1^{(1)} = 1 + \Gamma', \quad H_1^{(2)} = \frac{1}{A} + (1 + \Gamma')^2. \quad (2.21)$$

III. THE CONTINUUM INTENSITY: $\overline{y^2} - \bar{y}^2$

From (2.18) and (2.20) we get at once the general result for $v \geq 0$,

$$P_C \equiv \overline{y^2} - \bar{y}^2 = 2^v \Omega_{2A}^v \left\{ \frac{\Gamma(v+\frac{1}{2})}{2\pi^{\frac{1}{2}}} H_1^{(v)} - \left(\frac{\Gamma(\frac{v+1}{2})}{2\pi^{\frac{1}{2}}} H_1^{(v/2)} \right)^2 \right\}, \quad (2.22)$$

which is the generalization of [1; (13.7)], in the classical purely Gaussian cases, to the present, dominant non-Gaussian noise component $\Omega_{2A} (>> \sigma_G^2)$. In these classical cases, we can show at once that

$$\lim_{\Omega_{2A} \rightarrow 0} H_1^{(v)} \Omega_{2A}^v \rightarrow \lim_{\Omega \rightarrow 0} e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n\Omega}{A} + \sigma_G^2 \right)^v \rightarrow \sigma_G^{2v} (= \psi^v), \quad (2.23)$$

where $\Omega_{2A} \rightarrow \infty$ implies $A \rightarrow 0$ and $B_0^2 \rightarrow 0$, cf. (2.2b), so that (2.22) becomes, as expected,

$$P_C|_{\text{Gauss}} = 2^v \sigma_G^{2v} \left\{ \frac{\Gamma(v+\frac{1}{2})}{2\pi^{\frac{1}{2}}} - \frac{\Gamma^2(\frac{v+1}{2})}{4\pi} \right\} (> 0), \quad v \geq 0. \quad (2.24)$$

Figure 13.5 of [1] shows (2.24) as a function of rectifier law (v), as well as (2.18), (2.20) in these Gaussian cases. In the present, more general, situation of Class A noise, the results are more complex, as expected, with now two additional parameters (A, Γ'), descriptive of this much broader class of interference.

2.2-3: CASE A, FIGURE 2.1

We turn now to the more general problem of the covariance of the Class A non-Gaussian random field, sampled according to procedure (A), shown schematically in figure 2.1 earlier. Here, $x = \alpha(R, t)$, sensed at (R_1, t_1) , (R_2, t_2) , where $L = L$, cf. (3.3) in [3; (3.2)]. Equation (2.7) applies here, with $\Delta R \neq 0$ (as well as for $\Delta R = 0$), and we use (2.9a,b) for our illustrative examples, which are discussed in section 3 following. At this point, we recall from (2.3a) that the proper time delay to use is $\tau' = \tau - \Delta R/c_0$ in $\rho = \rho(\tau')$, and in some of the structural elements of the noise field covariances, cf. [3; (3.11b,c)].

CASE I: $\hat{r}' = 0$

From (2.7), we have $\rho = 1$, $m_1 = m_2 = 1$, giving

$$\hat{M}_y(\hat{\Delta R}, 0) = e^{-A} \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma' \right)^v B_v \Big|_{a=n}, \quad \rho = 1, \quad \therefore \tau = \frac{\Delta R}{c_0}, \quad (2.25)$$

where (2.7b) is specifically

$$y_{a=n} = \frac{\frac{n}{A} k_L(\hat{\Delta R}, 0) + \Gamma' k_G(\hat{\Delta R}, 0)}{\frac{n}{A} + \Gamma'}. \quad (2.25a)$$

For calculations, (2.9a,b) are used, with B_v given by (2.7a), where (2.25a) provides y_a . When $\hat{\Delta R} = 0$, (2.25) reduces to (2.15) et seq. for the total intensity of the field observed at $R_1 = R_2$.

CASE II: $\hat{\Delta R} \rightarrow \infty, |\tau'| > 1$

When $\hat{\Delta R} \rightarrow \infty$, we obtain different results, depending on τ' . Here $\rho = 0$, $Y_a \rightarrow 0$, cf. (2.9a,b) in (2.7b), and therefore $n = 0$. Accordingly, (2.7) becomes

$$\hat{M}_Y(\infty, |\tau'| > 1) = \hat{M}_Y(\infty, \infty) = \hat{M}_Y(0, \infty) = \bar{y}_{\text{norm}}^2, \quad (2.19) \quad (2.26)$$

The fact that $\hat{\Delta R} \rightarrow \infty$ ensures that $Y_a \rightarrow 0$, a behavior similar to that for Case (B) above, when we consider the purely Gaussian noise process, section 2.2-1.

CASE III: $\hat{\Delta R} \rightarrow \infty, 0 < |\tau'| < 1$

Here, $\rho > 0$ while $Y_a \rightarrow 0$, so that B_v , (2.7b), becomes $\Gamma^2(v+1/2)$ once more. The second moment function (2.7) is now

$$\begin{aligned} M_Y(\hat{\Delta R}, \tau') &= 1 \cdot \left(\frac{v+1}{2}\right) \exp[-A(2-\rho)] \sum_{m_1, m_2=0}^{\infty} \frac{[A(1-\rho)]^{m_1+m_2}}{m_1! m_2!} \\ &\times \sum_{n=0}^{\infty} \frac{(A\rho)^n}{n!} \left(\frac{n+m_1}{A} + \Gamma'\right)^{v/2} \left(\frac{n+m_2}{A} + \Gamma'\right)^{v/2}, \end{aligned} \quad (2.27)$$

which is a minor simplification of (2.7).

CASE IV: $\hat{\Delta R} \rightarrow \infty, |\tau'| = 0$

In this special situation, where $\tau = \Delta R/c_0 \rightarrow \infty$ in such a way that $\tau' = 0$ and therefore $\rho = 1$, $Y_a = 0$, we obtain directly from (2.27) the comparatively simple result,

$$\hat{M}_Y(\infty, 0) = \Gamma^2\left(\frac{v+1}{2}\right) H_1^{(v)} \quad (> 0). \quad (2.28)$$

2.2-4: REMARKS

At first glance, as $\Delta R \rightarrow \infty$, we might expect M_Y always to reduce to \bar{y}^2 , e.g., $K_Y \equiv M_Y - \bar{y}^2 = 0$ for the covariance of the rectified space-time field. This is expectedly the case for the covariance (and second-moment) function of the input Class A and Gauss noise field components $\alpha(R,t)$, as we can see directly from (2.9a,b), or from [3; (3.11b,c)] for example, in the physically derived cases. However, the process or field $y = g(x)$ here is the result of a nonlinear operation, cf. (2.5), (2.6), which severely distorts the input waveform and generates all kinds of modulation products, associated with the spatial as well as the temporal variations of the input field. This accounts for the departures in Cases III, IV of $M_Y(\infty, \tau')$ from \bar{y}^2 , while certainly $M_X(\infty, \tau') \rightarrow \bar{x}^2 = 0$, (since $\bar{x} = 0$ initially here).

From the various limiting results above, we see that

$$M_Y(0,0) > M_Y(\infty,0) \quad \text{and} \quad M_Y(0,0) > M_Y(0,\infty) , \quad (2.29a)$$

and

$$M_Y(\infty,0) \gtrless M_Y(0,\infty) \quad \text{depending on } A, \Gamma', \text{ and } v , \quad (2.29b)$$

with

$$M_Y(0,0) - M_Y(0,\infty) > 0 , \quad \text{cf. (2.19) and (2.20) ,} \quad (2.29c)$$

$$M_Y(\infty,\infty) = M_Y(0,\infty) = \bar{y}^2 , \quad \text{cf. (2.20) and (2.26) ,} \quad (2.29d)$$

whereas

$$M_X(0,0) > M_X(0,\infty) = M_X(\infty,0) = \bar{x}^2 = 0 . \quad (2.29e)$$

Finally, we note that (2.11), (2.12) apply here, also, for the Gauss-alone cases, where now

$$\Omega_{2A}(1 + \Gamma') \rightarrow \sigma_G^2 \quad \text{and} \quad B_v|_{a=0} \rightarrow (1 + \Gamma')^v B_v|_a = 0 .$$

2.2-5: SPECTRA

The various intensity spectra associated with the output of the processor (cf. figure 2.1) are important also, as they show how the energy in this output is distributed. Here, we consider two types of spectra, respectively, for the rectified spatial field (A) and for the process (B), namely the wavenumber and the frequency spectrum of $y(R,0)$ and $y(0,t)$. In particular, wavenumber spectra are useful in the analysis of spatially distributed phenomena, paralleling the analysis of time-dependent phenomena.

I: WAVENUMBER SPECTRUM

The wavenumber intensity spectrum is defined here by

$$W_2(k,0)_Y = W_2(k,\tau)_Y \Big|_{\tau=0} \equiv \iint_{\Delta R} M_Y(\Delta R,0) \exp(ik \cdot \Delta R) d(\Delta R) \quad (2.30a)$$

$$= 2\pi \int_0^\infty M_Y(\Delta R,0) J_0(k\Delta R) \Delta R d(\Delta R) = W_2(k,0)_Y, \quad (2.30b)$$

with

$$k = (k_x, k_y), \quad \Delta R = |\Delta R|, \quad k = |k| \quad (2.30c)$$

for these isotropic fields, where k is an (angular) vector wavenumber. Using the normalization of (2.8), we get, with $\hat{k} \equiv k\Delta_L$,

$$\hat{W}_2(\hat{k},0)_Y \equiv \frac{W_2(k,0)_Y}{\Delta_L^2 2^v \Omega^{2v}/4\pi} = 2\pi \int_0^\infty \hat{M}_Y(x,0) J_0(\hat{k}x) x dx \quad (2.31)$$

for the normalized wavenumber intensity spectrum.

Since $\hat{M}_y(\infty, 0)$ is nonvanishing, cf. (2.28), there is a dc component, or δ -function, in the general wavenumber spectrum. We will use the relations

$$\int_0^{\infty} x J_0(\hat{k}x) dx = \frac{1}{\hat{k}} \delta(\hat{k} - 0) , \quad \hat{k} = \left(\hat{k}_x^2 + \hat{k}_y^2 \right)^{\frac{1}{2}} = |\hat{k}| , \quad \hat{k} = (\hat{k}, \phi) , \quad (2.32)$$

where we must remember that \hat{k} is two-dimensional. With \hat{v} a vector wavenumber defined by

$$\hat{k} = 2\pi\hat{v} [= (\hat{v}, \phi)] , \quad \hat{k} = 2\pi\hat{v} = 2\pi|\hat{v}| , \quad (2.33a)$$

and using the relation

$$\delta(ax - b) = \frac{1}{a} \delta\left(x - \frac{b}{a}\right) \quad \text{for } a > 0 , \quad (2.33b)$$

we also show that

$$\begin{aligned} \delta(\hat{k}_x - 0) \delta(\hat{k}_y - 0) &= \frac{1}{2\pi\hat{k}} \delta(\hat{k} - 0) = \frac{1}{(2\pi)^3\hat{v}} \delta(\hat{v} - 0) \\ &= \frac{1}{(2\pi)^2} \delta(\hat{v}_x - 0) \delta(\hat{v}_y - 0) , \quad \hat{v} = |\hat{v}| , \quad \hat{v} = \left(\hat{v}_x^2 + \hat{v}_y^2 \right)^{\frac{1}{2}} . \end{aligned} \quad (2.34)$$

Applying (2.32) - (2.34), with

$$\begin{aligned} \hat{W}_2(\hat{k}, \phi)_y &= 2\pi \int_0^{\infty} x J_0(\hat{k}x) \left[\hat{M}_y(x, 0) - \hat{M}_y(\infty, 0) \right] dx \\ &\quad + 2\pi \hat{M}_y(\infty, 0) \int_0^{\infty} x J_0(\hat{k}x) dx \end{aligned} \quad (2.35a)$$

$$\begin{aligned} &= \hat{W}_2(\hat{k}, 0)_{y\text{-cont}} + (2\pi)^2 \hat{M}_y(\infty, 0) \delta(\hat{k}_x - 0) \delta(\hat{k}_y - 0) \\ &= \hat{W}_2(\hat{k}, 0)_{y\text{-cont}} + \hat{M}_y(\infty, 0) \delta(\hat{v}_x - 0) \delta(\hat{v}_y - 0) , \end{aligned} \quad (2.35b)$$

which defines $\hat{W}_2(\hat{k}, 0)_{y\text{-cont}}$, the continuous portion of the spectrum and shows the dc term in \hat{k} - or \hat{v} -space, as convenient. It is $\hat{W}_{2\text{-cont}}$ with which we are concerned in the specific numerical examples of section 3 ff.

II. FREQUENCY SPECTRUM

Here we employ the Wiener-Khintchine theorem [1; (3.42)] to write for the frequency spectrum of y

$$W_Y(f) = 2 \int_{-\infty}^{\infty} M_Y(0, \tau) \exp(-i\omega\tau) d\tau = B_0 \int_0^{\infty} \hat{M}_Y(0, \hat{\tau}) \cos(\hat{\omega}\hat{\tau}) d\hat{\tau}, \quad (2.36)$$

where

$$B_0 \equiv \Omega_{2A}^v 2^v / \pi\beta; \quad \hat{\tau} = \beta\tau; \quad \omega = 2\pi f; \quad \hat{\omega} \equiv \omega/\beta; \quad \therefore \hat{f} = f/\beta. \quad (2.36a)$$

Accordingly, we define the normalized frequency intensity spectrum of y as

$$\hat{W}_Y(\hat{f}) \equiv W_Y(f)/B_0 = \int_0^{\infty} \hat{M}_Y(0, \hat{\tau}) \cos(\hat{\omega}\hat{\tau}) d\hat{\tau}. \quad (2.37)$$

Again, there is a dc component, since $\hat{M}_Y(0, \infty) = \bar{y}^2 (> 0)$, cf. (2.20). We have

$$\hat{W}_Y(f) = \int_0^{\infty} [\hat{M}_Y(0, \hat{\tau}) - \hat{M}_Y(0, \infty)] \cos(\hat{\omega}\hat{\tau}) d\hat{\tau} + \frac{1}{2} \hat{M}_Y(0, \infty) \delta(\hat{f}-0), \quad (2.38)$$

since

$$\int_0^{\infty} \cos(\omega x) dx = \pi \delta(\omega-0) = \frac{1}{2} \delta(f-0).$$

As in the wavenumber cases above (Case I), we are concerned with the continuous part of the spectrum, viz.

$$\hat{W}_y(f)_{\text{cont}} = \int_0^{\infty} \left[\hat{M}_y(0, \hat{t}) - \bar{y}^2 \right] \cos(\omega \hat{t}) d\hat{t} , \quad (2.39)$$

which is also illustrated numerically in section 3 ff.

III. WAVENUMBER FREQUENCY SPECTRUM

The wavenumber frequency spectrum is defined by

$$W_2(k, \omega)_y = \iint_{-\infty}^{\infty} M_y(\Delta R, \tau) \exp(ik \cdot \Delta R - i\omega \tau) d(\Delta R) d\tau , \quad (2.40)$$

with $\omega = 2\pi f$. The associated wavenumber spectrum $W_2(k, 0)$ used in (2.30) is obtained from $W_2(k, \tau) \big|_{\tau=0}$. In normalized form, we have for (2.40), in these isotropic cases,

$$\begin{aligned} \hat{W}_2(\hat{k}, \hat{\omega})_y &= \left(2^v \Omega_{2A}^v \Delta_L^2 / (4\pi\beta) \right)^{-1} W_2(k, \omega)_y \\ &= \iint_{-\infty}^{\infty} \hat{M}_y(\hat{\Delta R}, \hat{t}) \exp(i\hat{k} \cdot \hat{\Delta R} - i\hat{\omega} \hat{t}) d(\hat{\Delta R}) d\hat{t} ; \end{aligned}$$

$$\hat{W}_2(\hat{k}, \hat{\omega})_y = 2\pi \int_0^{\infty} \int_0^{\infty} \hat{M}_y(x, \hat{t}) J_0(\hat{k}x) \exp(-i\hat{\omega} \hat{t}) x dx d\hat{t} . \quad (2.41)$$

The various dc components are readily extracted, as in Cases I and II above. Numerical examples of this joint intensity spectrum are reserved to a possible subsequent study. The results of section 3 show the marginal spectra (Cases I, II) of this more general situation.

2.2-6: FREQUENCY AND PHASE MODULATION BY CLASS A AND GAUSSIAN NOISE

This is a Case (B) situation, cf. figure 2.1, where $\Delta R = 0$ and we are concerned only with the received (non-Gaussian) noise process which is used to angle-modulate a (high frequency) carrier f_0 . For the analysis, see [3; section II].

The general result for the covariance of the carrier modulated by Class A and Gauss noise is found to be

$$K_Y(\tau)_{A+G} = \frac{1}{2} A_0^2 \operatorname{Re} \left[\exp \left(i \omega_0 \tau - D_0^2 \varrho(\tau)_G - A(2-\rho) \right. \right. \\ \left. \left. + 2A(1-\rho) \exp \left[-D_0^2 \varrho_0 / A \right] \right) \right], \quad (2.42)$$

where now, cf. [1; (4.2), (14.14c)],

$$\varrho(\tau)_G \Big|_{FM} = \left(\sigma_G^2 \text{ or } \frac{1}{A} \varrho_{2A} \right) \int_0^{|\tau|} (|\tau| - \lambda) k(\lambda) d\lambda \quad (D_0 = D_F) \\ = \left(1 \text{ or } \frac{1}{A} \right) \int_0^\infty W_{(G \text{ or } L)}(f) \frac{1 - \cos(\omega \tau)}{\omega^2} df. \quad (2.42a)$$

Also, cf. [1; (14.2), (14.14c)],

$$\varrho(\tau)_G \Big|_{PM} = \left(\sigma_G^2 \text{ or } \frac{1}{A} \varrho_{2A} \right) \{k(0) - k(\tau)\}_{(G \text{ or } L)} \quad (D_0 = D_P) \quad (2.42b)$$

and

$$\varrho_0 \rightarrow \varrho_0 \Big|_{FM} = \int_0^\infty W_A(f) df / \omega^2 \quad \text{or} \quad \varrho_0 \Big|_{PM} = \varrho_{2A}. \quad (2.42c)$$

For our numerical examples, we use the RC-spectrum of [1; section 14.1-3], where now

$$k_L(\zeta) = \exp(-b|\zeta|) , \quad k_G(\zeta) = \exp(-|\zeta|) , \quad \zeta = \tau \Delta\omega_N , \quad (2.43)$$

and therefore

$$\begin{aligned} \text{FM: } D_F^2 \varrho(\tau)_A &= \frac{1}{Ab^2} \left(\mu_F^2 \right)_A [\exp(-b|\zeta|) + b|\zeta| - 1] ; \\ D_F^2 \varrho(\tau)_G &= \Gamma' \left(\mu_F^2 \right)_A [\exp(-|\zeta|) + |\zeta| - 1] ; \\ \left(\mu_F^2 \right)_A &= \varrho_{2A} D_F^2 / \Delta\omega_N^2 ; \end{aligned} \quad (2.44a)$$

$$\begin{aligned} \text{PM: } D_P^2 \varrho(\tau)_A &= \frac{1}{A} \left(\mu_P^2 \right)_A [1 - \exp(-b|\zeta|)] ; \\ D_P^2 \varrho(\tau)_G &= \Gamma' \left(\mu_P^2 \right)_A [1 - \exp(-|\zeta|)] ; \\ \left(\mu_P^2 \right)_A &= D_P^2 \varrho_{2A} , \end{aligned} \quad (2.44b)$$

with $b (> 0)$ a dimensionless quantity, as is ζ . The quantity $\Delta\omega_N$ is the bandwidth of the modulating (Gauss) noise, cf. (2.43). Note, also, that

$$\Gamma' \left(\mu_F^2 \right)_A = \sigma_G^2 D_F^2 / \Delta\omega_N^2 = \left(\mu_F^2 \right)_G ; \quad \Gamma' \left(\mu_P^2 \right)_A = \sigma_G^2 D_P^2 = \left(\mu_P^2 \right)_G . \quad (2.45)$$

The quantities $(\mu_{F,P}^2)_{()}$ are the respective modulation indexes for FM and PM, cf. [1; chapter 14].

Finally, we have for ρ in (2.3), now with $\Delta R = 0$,

$$\rho(\tau) \rightarrow \rho(\zeta) = \begin{cases} 1 - \frac{\beta|\zeta|}{\Delta\omega_N} & \text{for } \frac{\beta|\zeta|}{\Delta\omega_N} \leq 1 \\ 0 & \text{for } \frac{\beta|\zeta|}{\Delta\omega_N} > 1 \end{cases} . \quad (2.46)$$

Putting the above (2.43), (2.44) in (2.42), we now specialize our results,

$$K_Y(\tau)_{A+G} = \frac{1}{2} A_O^2 k_O(\tau) \cos(\omega_O \tau) , \quad \text{with } k_O(0) = 1 , \quad (2.47)$$

to the normalized covariance $k_O(\tau)$, respectively, for FM and PM, and their associated spectra. We have for these carriers modulated by a sum of Gaussian and Class A noise:

I. FREQUENCY MODULATION

$$k_O(\zeta)_{FM} = \exp \left[-\Gamma' \left(\mu_F^2 \right)_A \left[\exp(-|\zeta|) + |\zeta| - 1 \right] - A(2-\rho) \right. \\ \left. + A\rho \exp \left(-\frac{1}{Ab^2} \left(\mu_F^2 \right)_A \left[\exp(-b|\zeta|) + b|\zeta| - 1 \right] \right) \right] , \quad (2.48)$$

with $\rho(\tau)$ given by (2.46). Here, $\varrho_O|_{FM} \rightarrow \infty$ in (2.42). Since

$$\lim_{\zeta \rightarrow \infty} k_O(\zeta)_{FM} = 0 ,$$

there is no dc in k_{O-FM} , and hence all the original carrier power ($\sim A_O^2/2$) is distributed into the sideband continuum for this highly nonlinear modulation, as expected [1; section 14.1-2].

The associated intensity spectrum for $k_O|_{FM}$ is defined by

$$W(\hat{\omega})_{A+G}|_{FM} = \int_0^\infty k_O(\zeta)_{FM} \cos(\hat{\omega}\zeta) d\zeta , \quad \hat{\omega} \equiv \frac{\omega - \omega_O}{\Delta\omega_N} , \quad (2.49)$$

which is determined by a direct cosine transform of $k_O(\zeta)_{FM}$. See appendix A.6 ff.

II. PHASE MODULATION

$$k_O(\zeta)_{PM} = \exp \left[-\Gamma' \left(\mu_P^2 \right)_A [1 - \exp(-|\zeta|)] + 2A(1-\rho) \exp \left(-\frac{1}{A} \left(\mu_P^2 \right)_A \right) \right. \\ \left. - A(2-\rho) + A\rho \exp \left(-\frac{1}{A} \left(\mu_P^2 \right)_A [1 - \exp(-b|\zeta|)] \right) \right] ,$$

(2.50)

with $\rho(\zeta)$ again given by (2.46). We note that

$$k_O(0)_{PM} = 1 , \quad (2.51a)$$

as before; that is, the total (normalized) intensity is unity. Also

$$k_O(\infty)_{PM} = \exp \left[-\Gamma' \left(\mu_P^2 \right)_A - 2A \left(1 - \exp \left(-\frac{1}{A} \left(\mu_P^2 \right)_A \right) \right) \right] : \quad (2.51b)$$

this is the fraction of the power remaining in the carrier, so that

$$k_O(0)_{PM} - k_O(\infty)_{PM} = 1 - (2.51b) , \quad (2.51c)$$

which is the fraction of the power distributed in the sideband continuum.

The associated intensity spectrum of the sideband continuum is determined from

$$w(\hat{\omega})_{A+G} \Big|_{PM-cont} = \int_0^\infty \left[k_O(\zeta)_{PM} - k_O(\infty)_{PM} \right] \cos(\hat{\omega}\zeta) d\zeta . \quad (2.52)$$

See section 3 ff. for examples and appendix A.5 for the evaluation methods.

Finally, in the equivalent Gaussian cases (Gauss noise modulation of equal intensity and basic spectrum, e.g.,

$$\Gamma' \left(\mu_F^2 \right)_A \rightarrow \Gamma' \left(\hat{\mu}_F^2 \right)_A = \Gamma' \mu_{FA}^2 (1 + \Gamma') \quad \text{and} \quad k_G \rightarrow k_L ,$$

we see that (2.48), (2.50) reduce to

$$k_O(\zeta)_{\text{FM-Gauss}} = \exp \left[- \left(\hat{\mu}_F^2 \right)_A \Gamma' [\exp(-b|\zeta|) + b|\zeta| - 1] / b^2 \right] ,$$

$$\left(\hat{\mu}_F^2 \right)_A = (1 + \Gamma') \left(\mu_F^2 \right)_A ; \quad (2.53a)$$

$$k_O(\zeta)_{\text{PM-Gauss}} = \exp \left[- \left(\hat{\mu}_P^2 \right)_A \Gamma' [1 - \exp(-b|\zeta|)] \right] ,$$

$$\left(\hat{\mu}_P^2 \right)_A = (1 + \Gamma') \left(\mu_P^2 \right)_A , \quad (2.53b)$$

with spectra obtained as before, from (2.49) and (2.52).

3. NUMERICAL ILLUSTRATIONS AND DISCUSSION

It is convenient to discuss the general results, namely the effects of (ZMNL) nonlinear rectifiers on, and modulation by, a mixture of Gaussian and non-Gaussian noise processes and fields, from the specific numerical calculations presented here in figures 3.1 - 3.10. These constitute a representative selection from the universe of possible parameter states [cf. "Summary of Normalized Parameters" and section 2, preceding]. This is done here on a per-figure basis, as noted below. In each case, the dc component is removed: only the covariance or continuous spectrum is calculated. We recall that there are two cases to distinguish: Case A, $\tau' = \tau - \Delta R/c_0$, a 2-element array; and Case B, $\tau' = \tau = t_2 - t_1$, a preformed beam. See figure 2.1 and (2.3a).

All spectra shown here are normalized to have area (under the spectrum level) of unity, i.e., the spectral normalization is obtained by dividing the spectrum by the value of the associated covariance at its origin. The normalization of the covariances themselves is obtained by dividing by the value at $\hat{t}=0$ or $\hat{\Delta R}=0$.

I. GAUSS NOISE ALONE

FIGURE 3.1

This figure shows the normalized temporal covariance ($\hat{\Delta R}=0$) for both the input and output of a ZMNL half-wave v -th law ($v \geq 0$) detector, when $v = 0, 1, 2$ and when only Gaussian noise ($A=0$) is applied to these nonlinear devices. These curves are based on (2.11) with (2.7a), where $Y_a = k_G$, (2.96), with $\Delta \hat{\omega}_G \equiv \Delta \omega_G / \beta = 5$ here. The normalization is with respect to the covariance maximum; e.g., the normalized covariance shown in figure 3.1 is obtained from $[(2.11)/(2.11)_{\hat{t}=0}]$, $\hat{\Delta R}=0$. These results apply for both cases A, B of figure 2.1, where now $\hat{t}' = \hat{t}$, since $\hat{\Delta R}=0$, cf. (2.3a) and remarks.

As expected (cf. [1; chapter 13]), the general nonlinearity (2.6), $v \geq 0$, contracts the covariance, which is equivalent to spreading the spectrum vis-à-vis the input, cf. figure 3.2, below. Moreover, the greater the distortion ($v=0,2$), usually the greater are these effects. [See appendix A.1.]

FIGURE 3.2

This is the same situation as shown in figure 3.1, except that the normalized intensity (frequency) spectrum is calculated now [cf. section 2.2-5, Case II, (2.39)] with $\hat{M}_y(0, \hat{f})$, (2.11), used in (2.39). Observe the greatly broadened spectra, particularly at the low spectral levels, where the greater spread occurs for the "super-clipper", $v=0$, cf. remarks, figure 3.1; also, appendix A.3.

FIGURE 3.3

For the same purely Gaussian field above, cf. (2.11) and (2.96), with $\hat{f}, \hat{f}'=0$, the spatial covariance is calculated, with parameters $\Delta_L/\Delta_G = 5^{1/2}$, using (2.11) as before. The normalization is with respect to the covariance at $\hat{\Delta R}=0$. Again, one observes the same kind of contraction in the covariance as noted in figure 3.1. [See appendix A.2.]

FIGURE 3.4

This is the wavenumber analogue of the frequency spectrum of figure 3.2, now with $\hat{f}', \hat{f}=0$, and is obtained from (2.35a,b) with $\Delta_L/\Delta_G = 5^{1/2}$. The rectification operation similarly spreads the wavenumber spectrum, with the greatest distortion ($v=0$) yielding the greatest wavenumber spread, as expected from the corresponding contraction of the associated covariance, cf. figure 3.3 above. [See appendix A.4.]

CLASS A PLUS GAUSS NOISE

FIGURE 3.5

The temporal covariance here is given by the general result (2.7), with the associated relations (2.7), (2.8), (2.9), wherein $\hat{\Delta R}=0$, so that $\hat{t}=\hat{t}'=t_2-t_1$, as before, and where $B_v|_a$, (2.7a), is now given analytically by (2.10) for $v = 0,1,2$. Here, the parameter values are $\Delta\hat{\omega}_G \equiv \Delta\omega_G/\beta = 5^{1/2}$, as before, now with $A=0.2$, $r'_A=10^{-3}$, $\Delta\omega_L/\beta \equiv \Delta\hat{\omega}_L = 1$ for the Class A non-Gaussian noise component, typically.

Again, for the super-clipper ($v=0$), the contraction in the normalized covariance is greatest, cf. figure 3.1. But the contribution of the comparatively strong non-Gaussian component exaggerates this effect. [See appendix A.1.]

FIGURE 3.6

The corresponding intensity (frequency) spectrum ($\hat{\Delta R}=0$), obtained from (2.7) in (2.39), however, shows a fine-structure not exhibited when Gauss noise alone ($A=0$) is applied to these ZMNL devices. The spectral levels for the case $v=0$, ($A=0$) and ($A>0$), cf. figure 3.2 with figure 3.6, are approximately the same, whereas the other inputs, cases $v=1,2$, are much elevated as f becomes larger, again due to the presence of the structured Class A noise, when $\beta\bar{T}_S \leq 1$, cf. (2.3): on the average, the original Class A "signals" are of comparatively short duration, or spectrally wide to begin with, so that clipping further spreads the spectrum. [See appendix A.3.]

FIGURE 3.7

The spatial covariance when Class A noise is added to the Gaussian input shows analogous behavior, cf. figures 3.3 and 3.5: the covariance is compressed vis-à-vis the input, but more so than in the Gauss-alone situations. Again, (2.7) - (2.10) are employed. [See appendix A.2.]

FIGURE 3.8

The corresponding wavenumber (intensity) spectrum with Class A noise and the Gaussian component, obtained from (2.7) - (2.10) in (2.39), is shown here. Comparison with figure 3.4 indicates a broader spectral input, due to the non-Gaussian component, but a relatively narrower output, although the latter is still noticeably spread vis-à-vis the original input. [See appendix A.4.]

FIGURE 3.9

Finally, we consider the angle-modulation cases described in section 2.2-6 above, where weak to strong angle modulations ($\mu \sim 1$ to 50) by Class A noise, with a weak ($\Gamma' = 10^{-3}$) Gaussian modulation component, is employed.

For phase modulation by non-Gaussian noise, based on (2.50) with (2.44b), (2.45), (2.46), the resulting normalized intensity (frequency) spectra are obtained by applying (2.50) to (2.52), where $f = \hat{\omega}/2\pi$; $\hat{\omega} = (\omega - \omega_0)/\Delta\omega_N$, cf. (2.49). Note the "spike" at $f \sim 0.1$, followed by a variety of sidelobes which rise as the phase modulation index μ_p increases. The spike is now bounded at $f \approx 0.8$, at the -10 dB level, when $\mu_p = 50$. As expected, the larger indexes (μ_p) produce broader spectra. [See appendix A.5.]

FIGURE 3.10

For frequency modulation by non-Gaussian noise, from (2.48) with (2.49) and (2.44a), the corresponding intensity (frequency) spectra again exhibit a continuous spike ($f < 0.1$). With small modulation indexes (μ_F), the spectra are less broad than for the larger indexes, as expected. The non-Gaussian noise component dominates the spectrum here. [See appendix A.6.]

EXTENSIONS

Other situations where the second-order Class A probability density functions may be applied are noted in [2] and [3]. We list some of the extensions of the analysis to the following "classical" problems:

- 1) The inclusion of representative signals, with Gauss and non-Gauss (Class A) noise, in the problems already treated here (section 2);
- 2) The case of the full-wave square-law rectifier, with both Class A and B noise, as well as Gauss noise;
- 3) The extension of 2) to include general broadband and narrowband signals;
- 4) The calculation of signal-to-noise ratios and deflection criteria, cf. [1; section 5.3-4].
- 5) Covariances and spectra for ZMNL system outputs, with signals as well as non-Gaussian noise inputs;
- 6) The rôle of the electromagnetic (or acoustic) interference (EMI or ACI) scenario, cf. [5; section 2B,5];
- 7) Evaluation of the large (FM,PM) indexes, or asymptotically Gaussian cases, cf. [12].

Further opportunities to extend the classical theory [2],[3], now with non-Gaussian noise inputs, are evident from the examples and methods described in [1; chapters 5, 12 - 16], for instance.

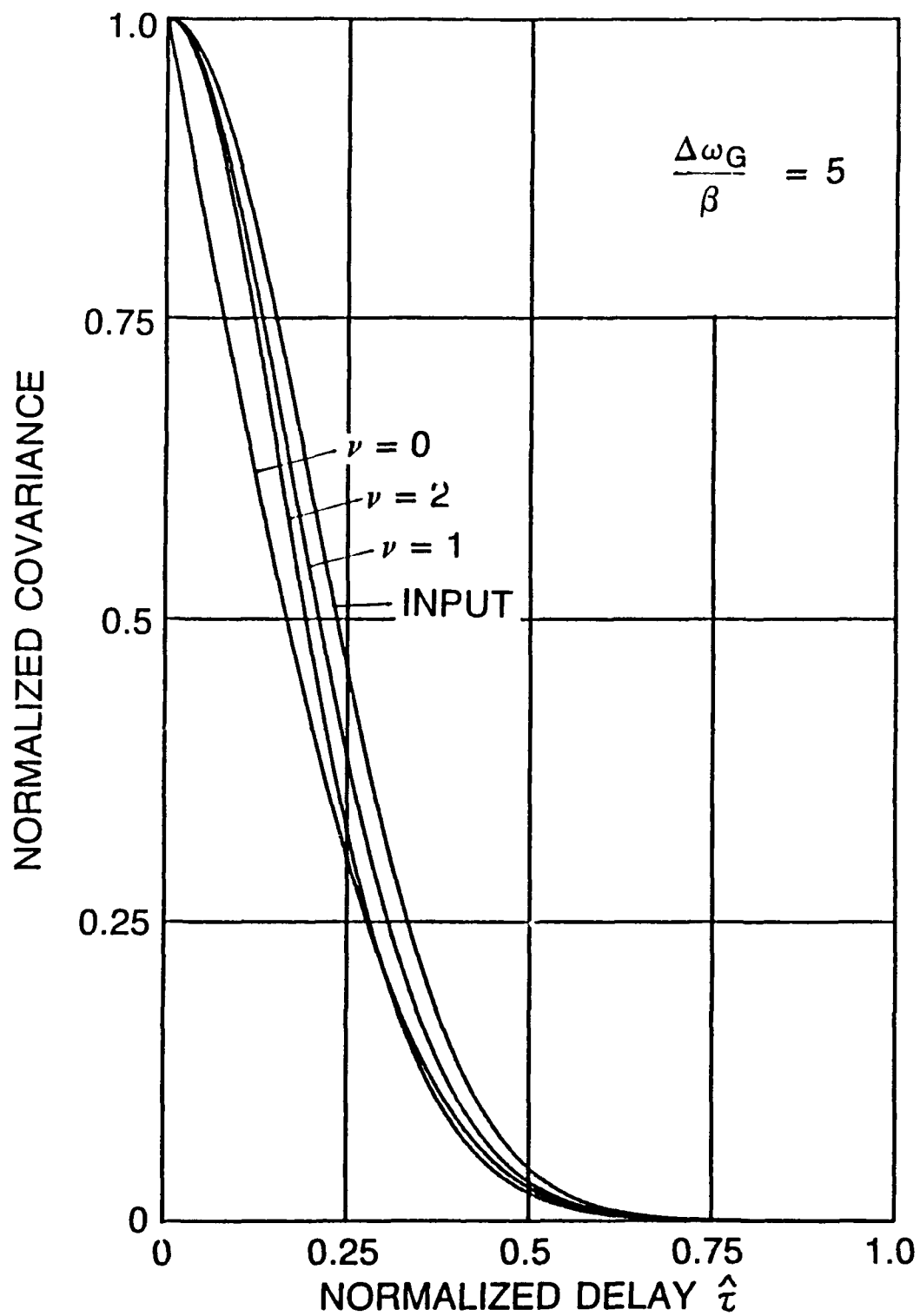


FIGURE 3.1 TEMPORAL COVARIANCE (FOR $\hat{\Delta R}=0$); GAUSS NOISE ONLY;
CF. (2.11) WITH (2.7a), (2.9b), AND APPENDIX A.1

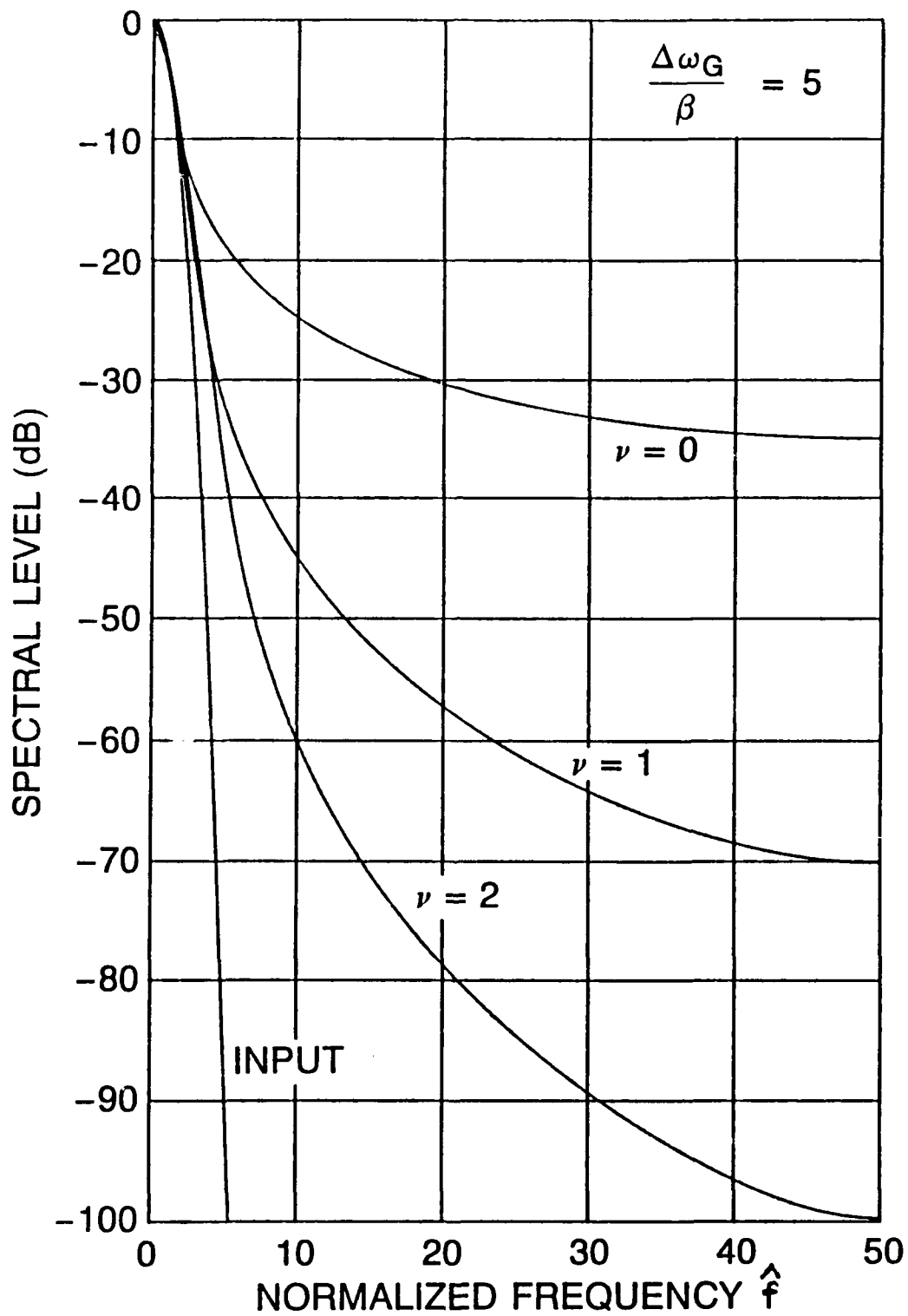


FIGURE 3.2 FREQUENCY (INTENSITY) SPECTRUM (FOR $\hat{\Delta R}=0$); GAUSS NOISE ONLY; CF. (2.39), USED WITH (2.11), AND APPENDIX A.3

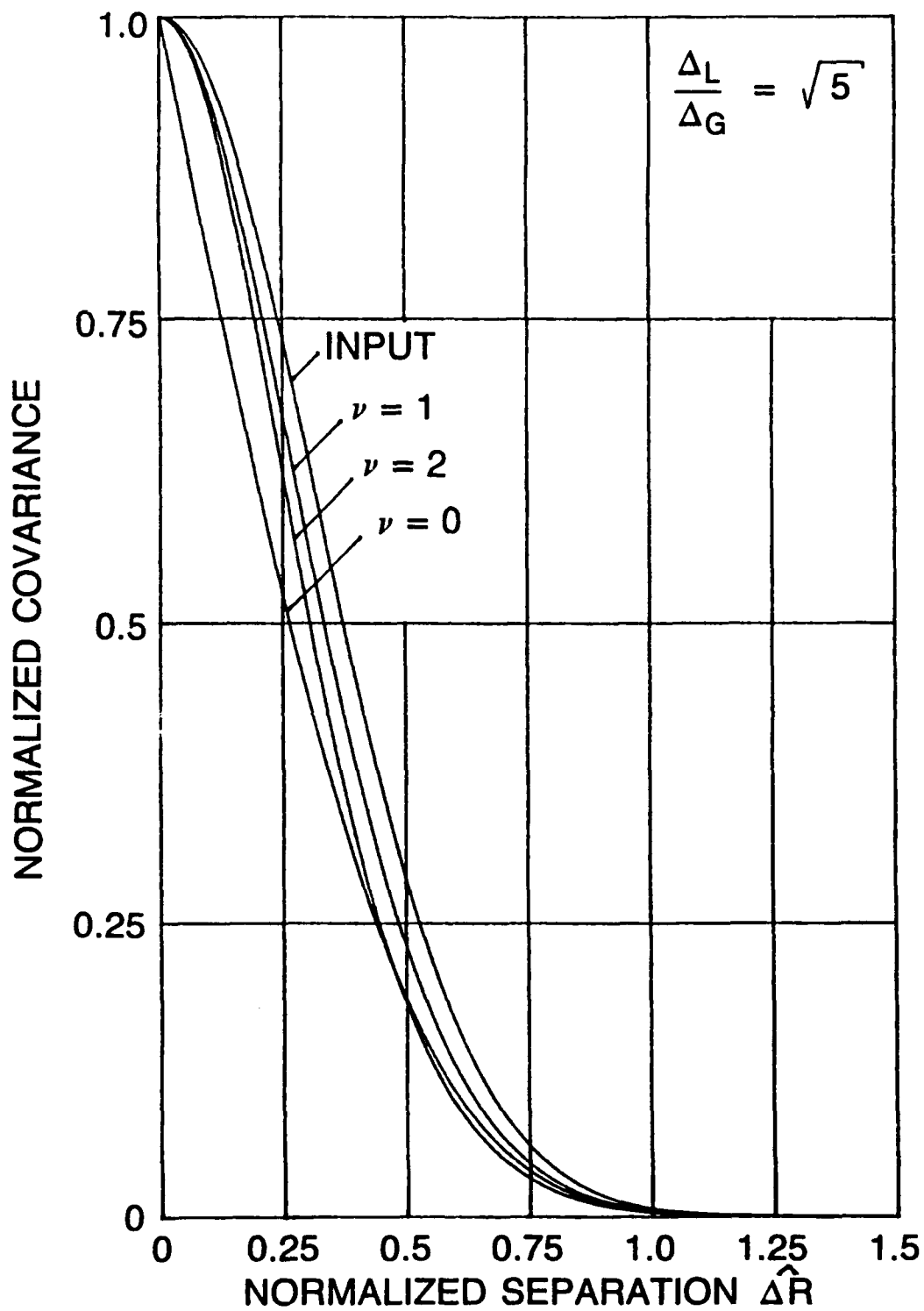


FIGURE 3.3 SPATIAL COVARIANCE (FOR $\hat{t}', \hat{t}=0$); GAUSS NOISE ONLY; CF. (2.11), (2.7b), AND APPENDIX A.2

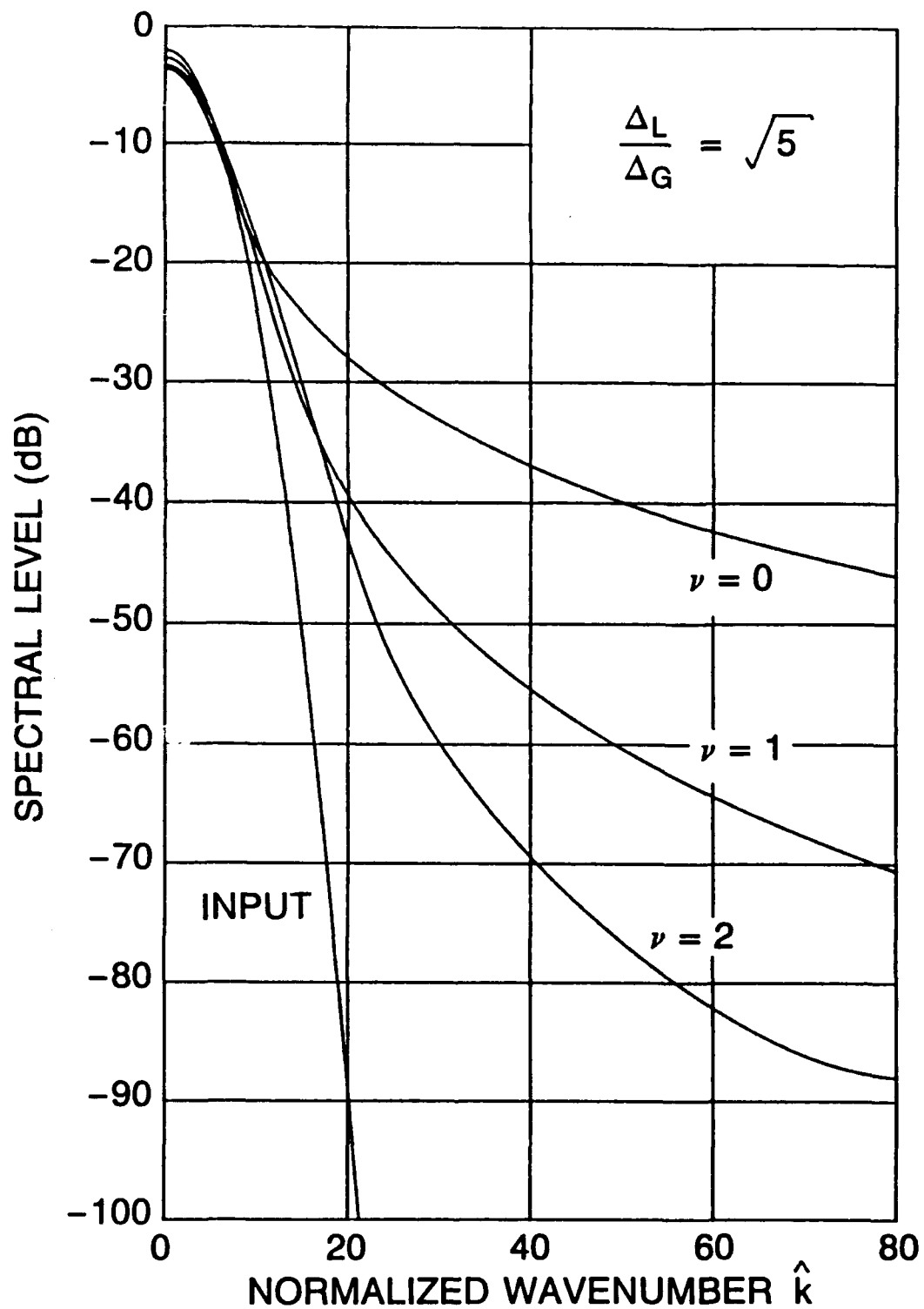


FIGURE 3.4 WAVENUMBER (INTENSITY) SPECTRUM (FOR $\hat{f}', \hat{f}=0$); GAUSS NOISE ONLY; CF. (2.35a,b) WITH (2.11), (2.7b), AND APPENDIX A.4

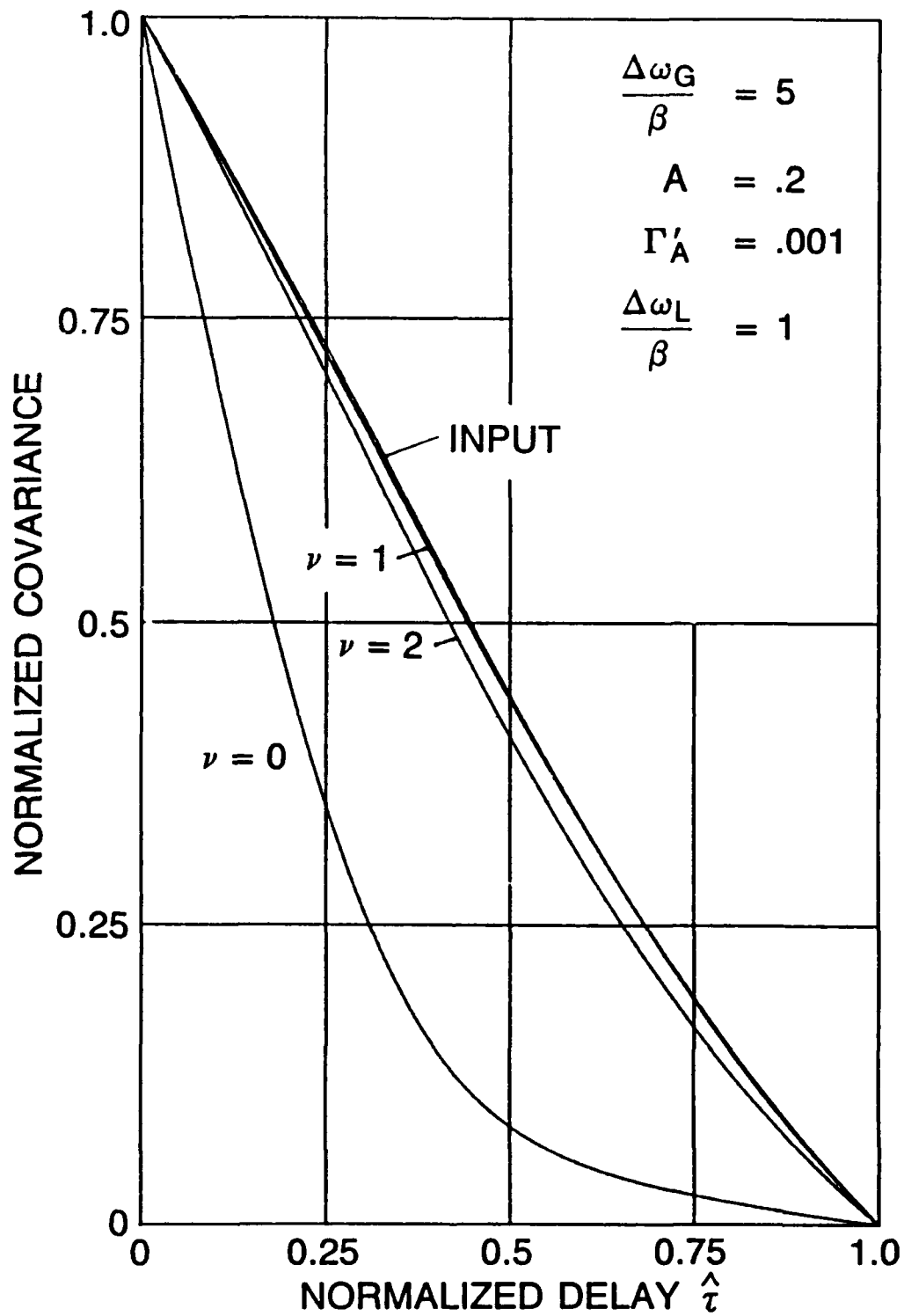


FIGURE 3.5 TEMPORAL COVARIANCE (FOR $\hat{\Delta R}=0$); CLASS A AND GAUSS NOISE; CF. (2.7)-(2.9) AND APPENDIX A.1

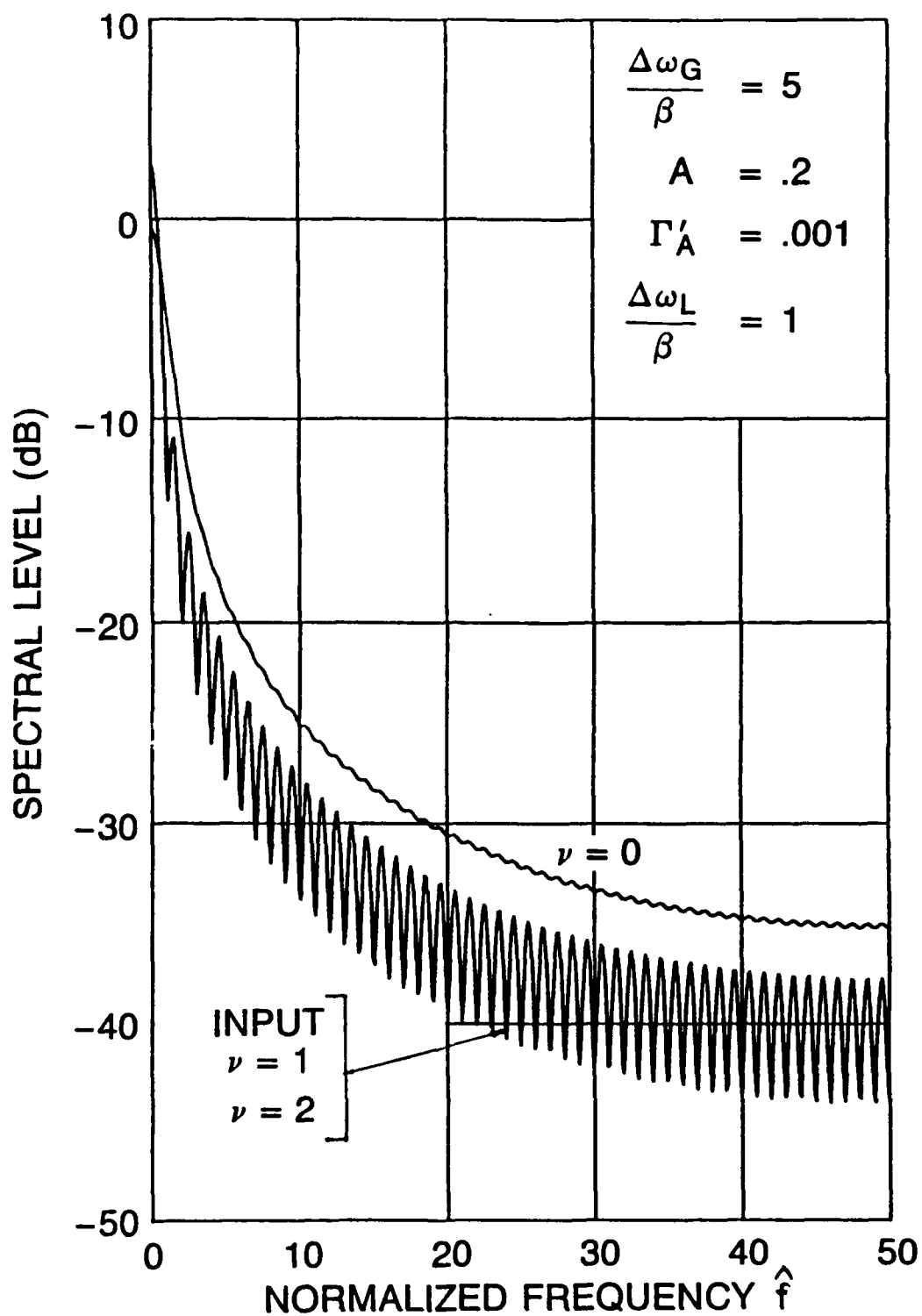


FIGURE 3.6 FREQUENCY (INTENSITY) SPECTRUM (FOR $\hat{\Delta R}=0$); CLASS A AND GAUSS NOISE; CF. (2.7) IN (2.39) WITH APPENDIX A.3

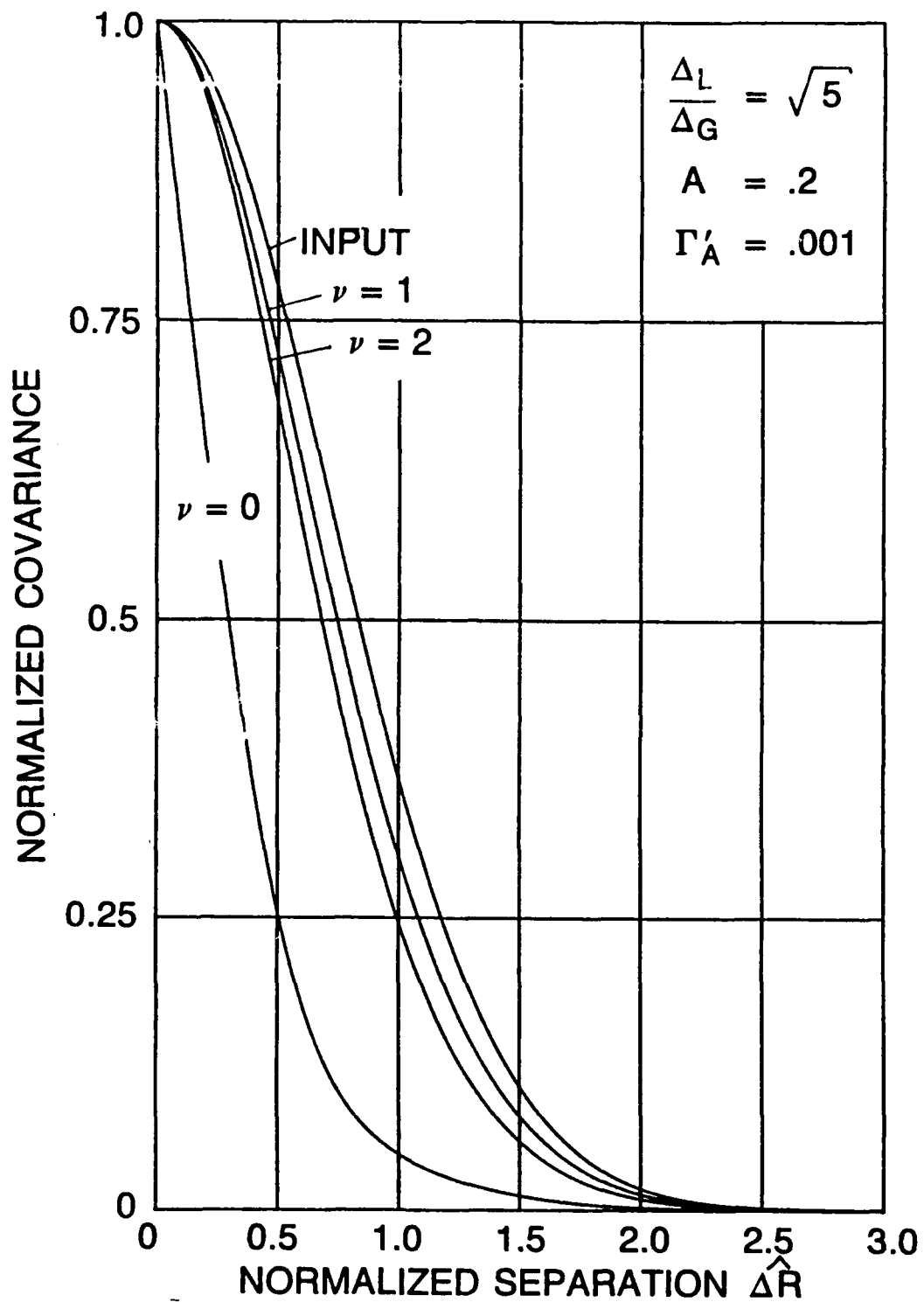


FIGURE 3.7 SPATIAL COVARIANCE (FOR $\hat{r}', \hat{r}=0$); CLASS A AND GAUSS NOISE; CF. (2.7)-(2.10) WITH APPENDIX A.2

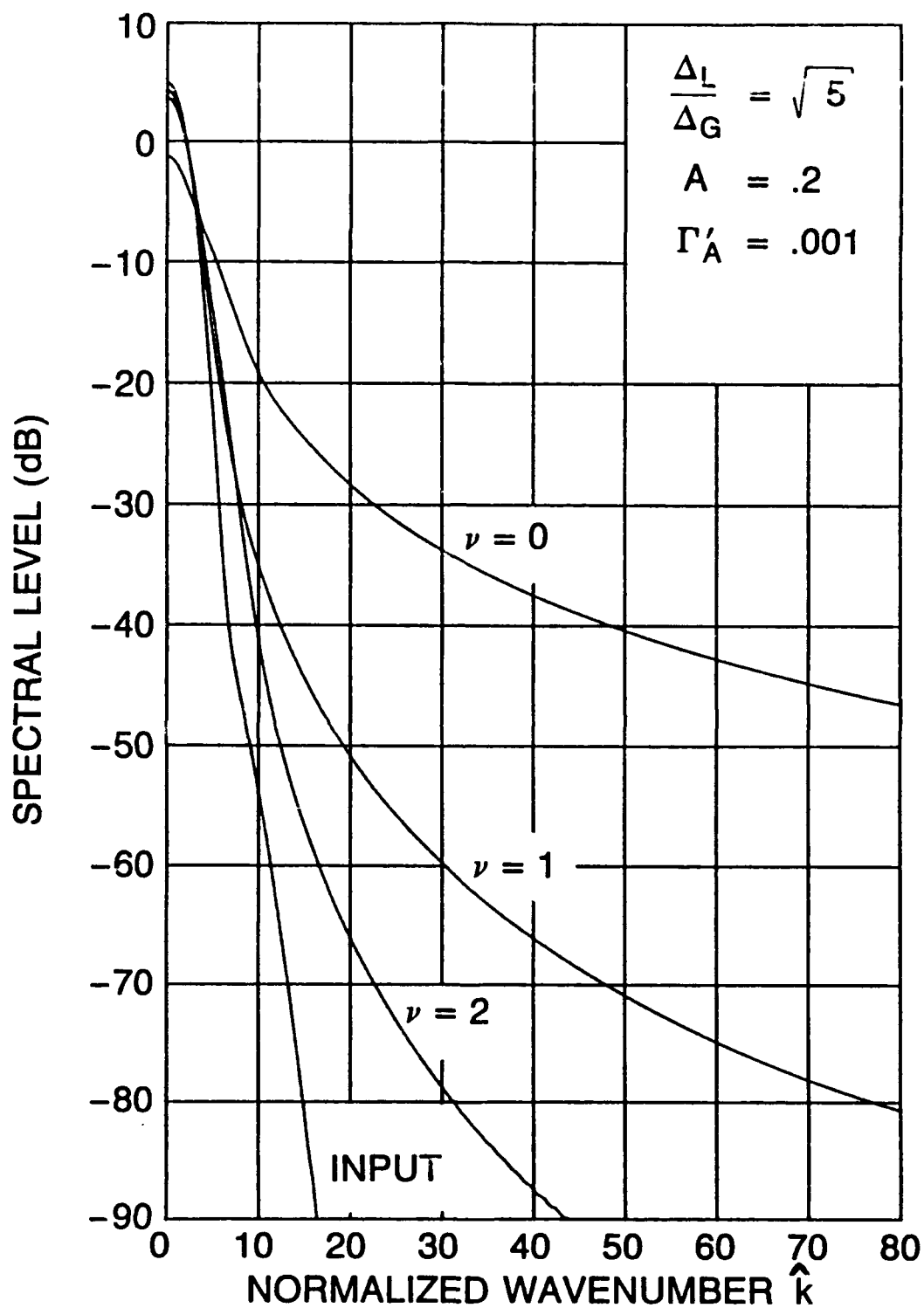


FIGURE 3.8 WAVENUMBER SPECTRUM (FOR $\hat{r}', \hat{r}=0$); CLASS A AND GAUSS NOISE; CF. (2.7)-(2.10) IN (2.38a,b) AND APPENDIX A.4

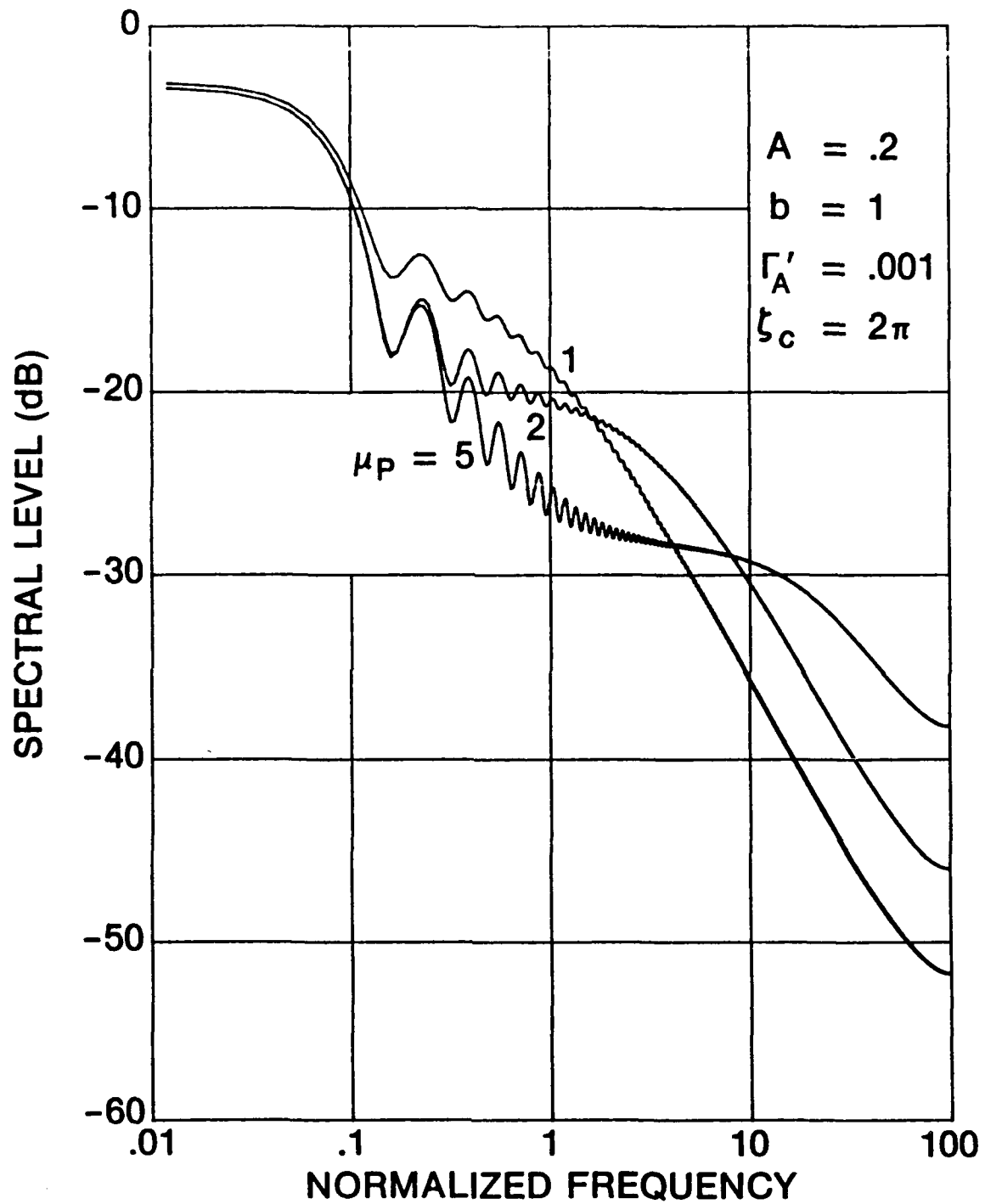


FIGURE 3.9a PHASE MODULATION (INTENSITY) SPECTRUM FOR INDEX $\mu_P=1,2,5$, CLASS A AND GAUSS NOISE; CF. (2.50) WITH (2.44b), (2.45), (2.46) IN (2.52), AND APPENDIX A.5

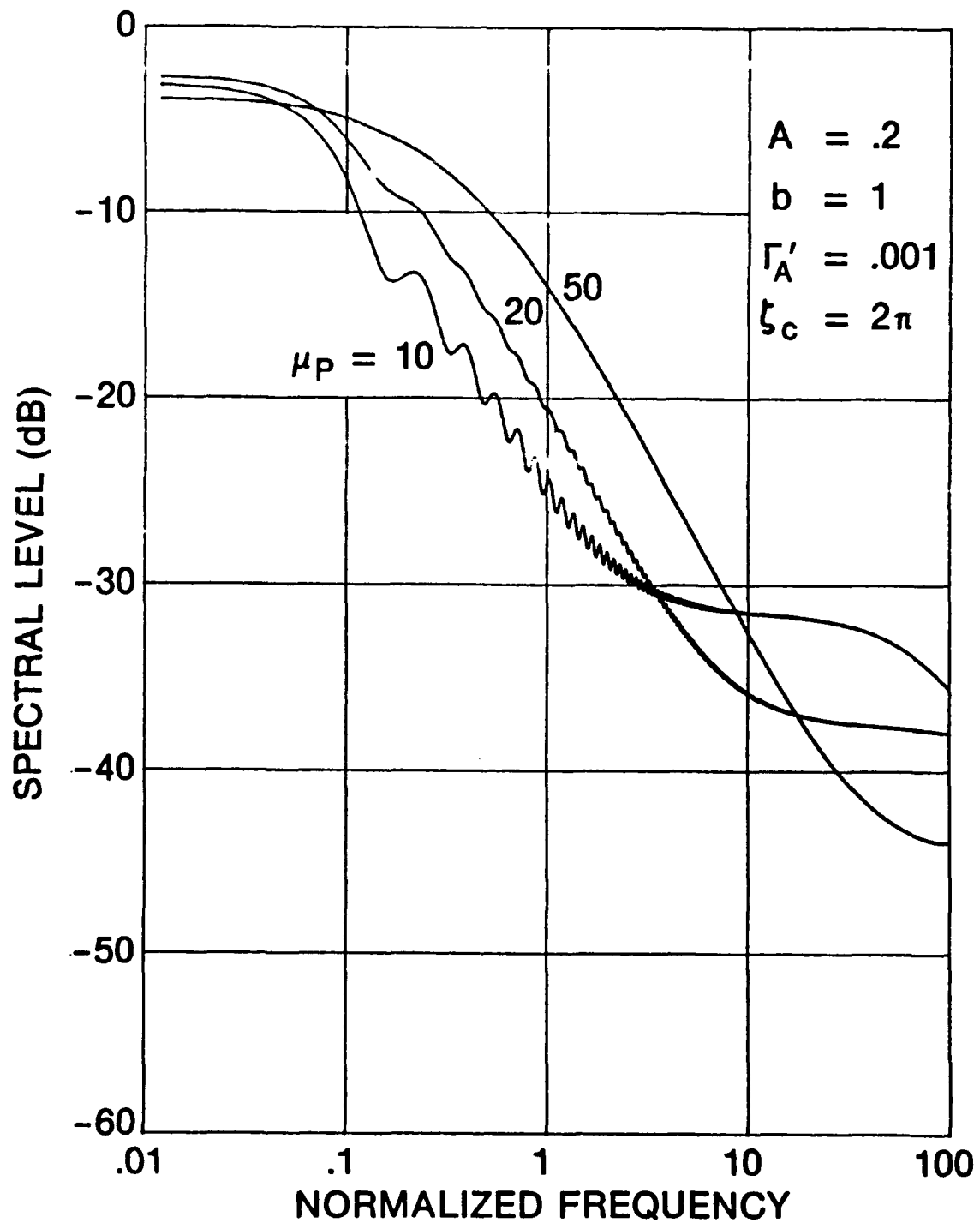


FIGURE 3.9b PHASE MODULATION (INTENSITY) SPECTRUM FOR INDEX $\mu_P=10,20,50$, CLASS A AND GAUSS NOISE; CF. (2.50) WITH (2.44b), (2.45), (2.46) IN (2.52), AND APPENDIX A.5

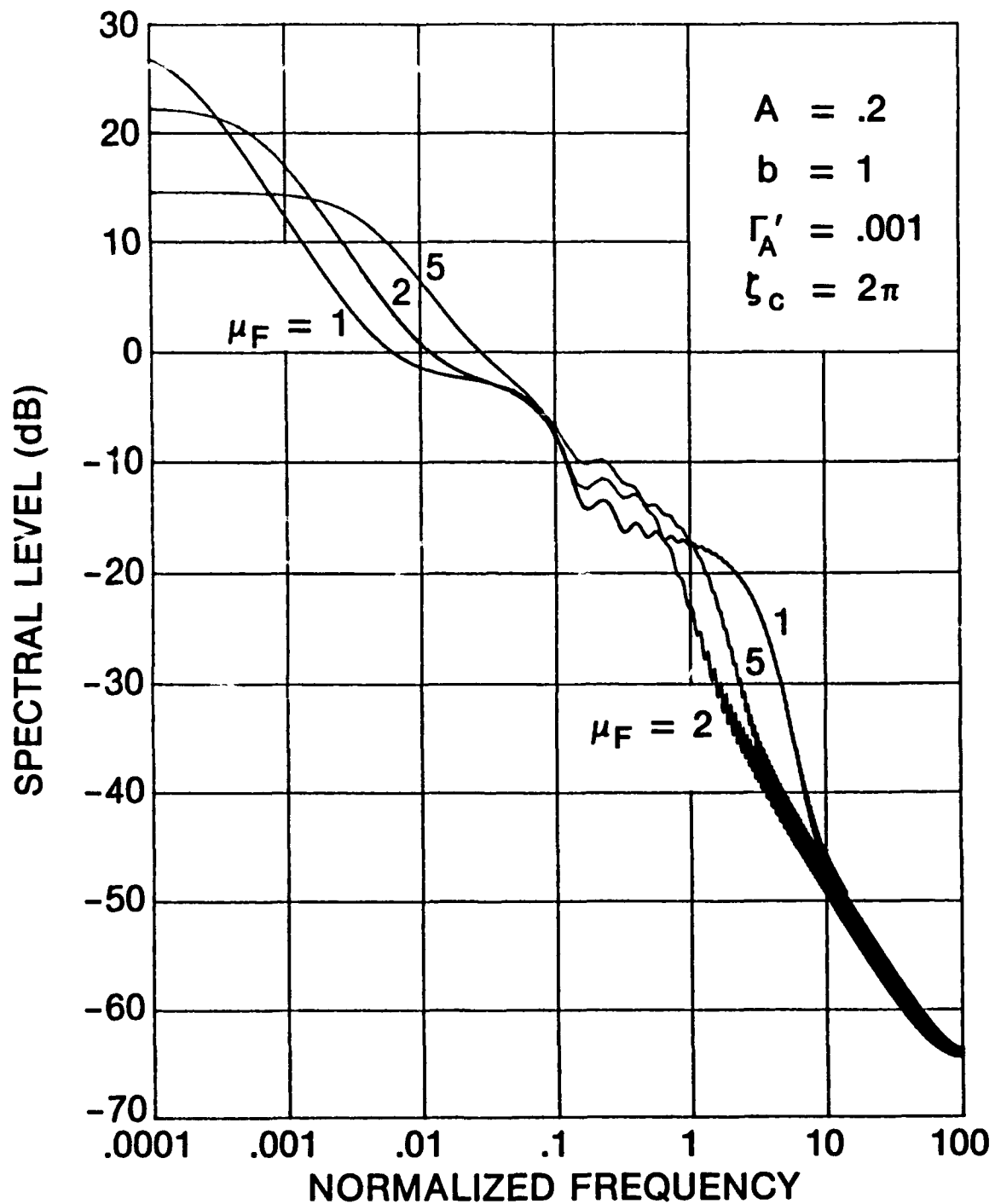


FIGURE 3.10a FREQUENCY MODULATION (INTENSITY) SPECTRUM
FOR INDEX $\mu_F=1,2,5$, CLASS A AND GAUSS NOISE;
CF. (2.48) WITH (2.49), (2.44a), AND APPENDIX A.6

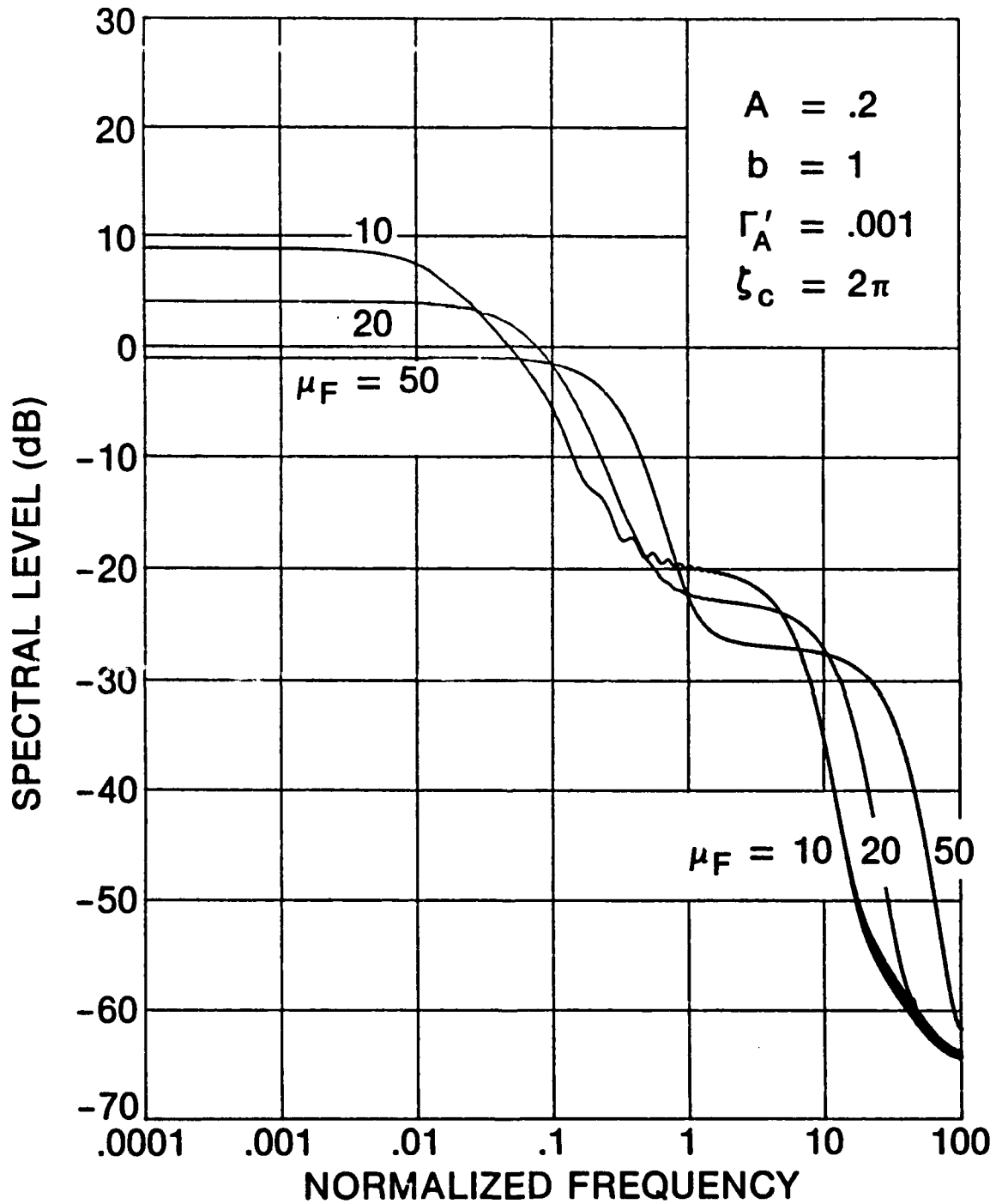


FIGURE 3.10b FREQUENCY MODULATION (INTENSITY) SPECTRUM
 FOR INDEX $\mu_F = 10, 20, 50$, CLASS A AND GAUSS NOISE;
 CF. (2.48) WITH (2.49), (2.44a), AND APPENDIX A.6

PART II. MATHEMATICAL AND COMPUTATIONAL PROCEDURES

4. SOME PROPERTIES OF THE COVARIANCE FUNCTION

In this section, we collect some useful relations for the covariance and auxiliary functions encountered in the numerical evaluation. These are necessary for rapid computation of the multiple series involved here and also serve as checks on the numerical procedures employed.

4.1 SIMPLIFICATION AND EVALUATION OF $B_v(Y)$

The function $B_v(Y)$ is defined by the following combination of hypergeometric functions:

$$B_v(Y) = \Gamma^2\left(\frac{v+1}{2}\right) F\left(-\frac{v}{2}, -\frac{v}{2}; \frac{1}{2}; Y^2\right) + \\ + 2 \Gamma^2\left(\frac{v}{2} + 1\right) Y F\left(\frac{1-v}{2}, \frac{1-v}{2}; \frac{3}{2}; Y^2\right) \quad \text{for } Y^2 \leq 1. \quad (4.1)$$

For the upper F function in (4.1), we have [1; (A.1.39b)]

$$v = 0, \quad F\left(0, 0; \frac{1}{2}; Y^2\right) = 1; \\ v = 1, \quad F\left(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; Y^2\right) = Y \arcsin(Y) + \left(1 - Y^2\right)^{\frac{1}{2}}; \\ v = 2, \quad F\left(-1, -1; \frac{1}{2}; Y^2\right) = 1 + 2Y^2; \quad (4.2)$$

where \arcsin is the principal value inverse sine function. On the other hand, for the latter F function in (4.1), we have [1; (A.1.39a) and (A.1.39c)]

$$v = 0, \quad F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; Y^2\right) = \frac{\arcsin(Y)}{Y} ;$$

$$v = 1, \quad F\left(0, 0; \frac{3}{2}; Y^2\right) = 1 ;$$

$$v = 2, \quad F\left(-\frac{1}{2}, -\frac{1}{2}; \frac{3}{2}; Y^2\right) = \frac{3}{4}\left(1 - Y^2\right)^{\frac{1}{2}} + \frac{1 + 2Y^2}{4Y} \arcsin(Y). \quad (4.3)$$

When these quantities are substituted in the above expression for $B_v(Y)$, we find the following relatively simple relations:

$$B_0(Y) = \pi + 2 \arcsin(Y) ,$$

$$B_1(Y) = Y \arcsin(Y) + \left(1 - Y^2\right)^{\frac{1}{2}} + \frac{\pi}{2}Y ,$$

$$B_2(Y) = \left(\frac{1}{2} + Y^2\right) \left(\frac{\pi}{2} + \arcsin(Y)\right) + \frac{3}{2}Y \left(1 - Y^2\right)^{\frac{1}{2}} . \quad (4.4)$$

These three quantities can be computed simultaneously by the following very compact computer coding in BASIC:

```

Y2=Y*Y
Sq=SQR(1.-Y2)
T=ASN(Y)+1.5707963267948966
B0=T+T
B1=Y*T+Sq
B2=(.5+Y2)*T+1.5*Y*Sq

```

(4.5)

Thus, the rather formidable expression, above, for $B_v(Y)$ can be evaluated by the use of just one square root and one arcsin when $v = 0, 1, 2$.

The following limiting values, which are obvious, are needed for various special cases:

$$\begin{aligned}
B_0(0) &= \pi, & B_0(1) &= 2\pi, \\
B_1(0) &= 1, & B_1(1) &= \pi, \\
B_2(0) &= \pi/4, & B_2(1) &= 3\pi/2, \\
B_3(0) &= 1, & B_3(1) &= 15\pi/4, \\
B_4(0) &= 9\pi/16, & B_4(1) &= 105\pi/8.
\end{aligned} \tag{4.6}$$

These are special cases of

$$B_\nu(0) = \Gamma^2\left(\frac{\nu+1}{2}\right), \tag{4.7}$$

$$B_\nu(1) = 2\pi^{1/2} \Gamma\left(\nu + \frac{1}{2}\right), \tag{4.8}$$

the latter following from [10; (15.1.20)].

4.2 LIMITING VALUES OF THE COVARIANCE FUNCTION

The covariance function at normalized separation $\hat{\Delta R}$ and delay \hat{t} is given by (2.7) as

$$\begin{aligned}
\hat{M}_Y(\hat{\Delta R}, \hat{t}) &= \exp[-A(2-\rho)] \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{[A(1-\rho)]^{m_1+m_2}}{m_1! m_2!} \\
&\times \sum_{n=0}^{\infty} \frac{(A\rho)^n}{n!} \left(\frac{n+m_1}{A} + \Gamma'_A\right)^{\nu/2} \left(\frac{n+m_2}{A} + \Gamma'_A\right)^{\nu/2} B_\nu(Y), \tag{4.9}
\end{aligned}$$

where

$$\rho = \rho(\hat{t}) = \max\{0, 1 - |\hat{t}|\}, \tag{4.10}$$

$$Y = Y(m_1, m_2, n) = \frac{\frac{n}{A} k_L + \Gamma'_A k_G}{\left(\frac{n+m_1}{A} + \Gamma'_A\right)^{1/2} \left(\frac{n+m_2}{A} + \Gamma'_A\right)^{1/2}}, \tag{4.11}$$

$$k_L = k_L(\hat{\Delta R}, \hat{r}) = \exp \left[-\hat{\Delta R}^2 - \frac{1}{2} \left(\frac{\Delta \omega_L}{\beta} \right)^2 \hat{r}^2 \right], \quad (4.12)$$

$$k_G = k_G(\hat{\Delta R}, \hat{r}) = \exp \left[- \left(\frac{\Delta_L}{\Delta_G} \right)^2 \hat{\Delta R}^2 - \frac{1}{2} \left(\frac{\Delta \omega_G}{\beta} \right)^2 \hat{r}^2 \right]. \quad (4.13)$$

The functions ρ , k_L , k_G can be replaced by other functional dependencies, if desired. The function $B_v(Y)$ has been considered earlier and considerably simplified for $v = 0, 1, 2$.

4.3 VALUE AT INFINITY

As $\hat{\Delta R}$ or $\hat{r} \rightarrow \pm\infty$, then

$$\rho \rightarrow 0, \quad k_L \rightarrow 0, \quad k_G \rightarrow 0, \quad Y \rightarrow 0. \quad (4.14)$$

(If $|\hat{r}|$ remains less than 1 as $\hat{\Delta R}$ tends to infinity, then ρ does not approach zero; this nuance has been discussed elsewhere in this report.) Then, it follows that

$$\begin{aligned} \hat{M}_Y &\rightarrow \exp(-2A) \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{A^{m_1+m_2}}{m_1! m_2!} \left(\frac{m_1}{A} + \Gamma'_A \right)^{v/2} \left(\frac{m_2}{A} + \Gamma'_A \right)^{v/2} B_v(0) \\ &= B_v(0) \left(\exp(-A) \sum_{m=0}^{\infty} \frac{A^m}{m!} \left(\frac{m}{A} + \Gamma'_A \right)^{v/2} \right)^2, \end{aligned} \quad (4.15)$$

because the sum on n can be terminated with the $n = 0$ term.

The sum on m can be effected in closed form, for $v = 0, 2, 4$, etc., by using the following results:

$$\sum_{m=0}^{\infty} \frac{A^m}{m!} = \exp(A), \quad (4.16)$$

$$\sum_{m=0}^{\infty} \frac{A^m}{m!} m = \sum_{m=1}^{\infty} \frac{A^m}{(m-1)!} = A \exp(A) , \quad (4.17)$$

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{A^m}{m!} m^2 &= \sum_{m=1}^{\infty} \frac{A^m}{(m-1)!} (m-1+1) \\ &= \sum_{m=2}^{\infty} \frac{A^m}{(m-2)!} + \sum_{m=1}^{\infty} \frac{A^m}{(m-1)!} = (A^2 + A) \exp(A) . \end{aligned} \quad (4.18)$$

There follows

$$\hat{M}_Y(\infty) = \begin{cases} \pi & \text{for } v = 0 \\ \frac{\pi}{4} (1 + \Gamma'_A)^2 & \text{for } v = 2 \\ \frac{9\pi}{16} \left[\frac{1}{A} + (1 + \Gamma'_A)^2 \right]^2 & \text{for } v = 4 \end{cases} . \quad (4.19)$$

The case for $v = 1$ requires a numerical summation, once A and Γ'_A are specified. When these limiting values are subtracted from the correlation function, we obtain the covariance function.

4.4 VALUE AT THE ORIGIN

For $\hat{\Delta R} = 0$, $\hat{t} = 0$, then

$$\rho = 1, \quad k_L = 1, \quad k_G = 1, \quad (4.20)$$

and

$$\hat{M}_Y(0,0) = \exp(-A) \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n}{A} + \Gamma'_A \right)^v B_v(1), \quad (4.21)$$

because the sums on m_1 and m_2 can be terminated with the zero terms, thereby also leading to $Y = 1$.

The sum on n can be accomplished in closed form, for $v = 0, 1, 2$, etc., by using results given earlier. There follows

$$\hat{M}_Y(0,0) = \left\{ \begin{array}{ll} 2\pi & \text{for } v = 0 \\ \pi \left(1 + \Gamma'_A \right) & \text{for } v = 1 \\ \frac{3}{2}\pi \left[\frac{1}{A} + \left(1 + \Gamma'_A \right)^2 \right] & \text{for } v = 2 \end{array} \right\}. \quad (4.22)$$

PART III. APPENDICES AND PROGRAMS

APPENDIX A.1 - EVALUATION OF COVARIANCE FUNCTION
FOR ZERO SEPARATION ($\hat{\Delta R} = 0$)

A program for the numerical evaluation of covariance $\hat{M}_y(\hat{\Delta R}, \hat{r})$ for $\hat{\Delta R} = 0$ is contained in this appendix. Inputs required of the user are A , Γ'_A , $(\Delta\omega_L/\beta)^2$, $(\Delta\omega_G/\beta)^2$, $\delta(\hat{r})$, $N(\hat{r})$, in lines 20 - 70. Since we are generally interested in values of A less than 1, the series for \hat{M}_y in (4.9) will not have to be taken to very large values of m_1 , m_2 , n ; accordingly, the values of $\{A^k/k!\}$ are tabulated once in lines 260 - 300 with a tolerance of $1E-10$ set in line 80.

The values of the covariance at infinity, as given by (4.19), are computed and subtracted in lines 220 - 240 and 400 - 420; this is in anticipation of taking a Fourier transform of a covariance function which decays to zero for large arguments $\hat{\Delta R}$.

The functions $B_v(Y)$ and $\hat{M}_y(\hat{\Delta R}, \hat{r})$ are available in the two subroutines starting at lines 1010 and 1120, respectively. The latter subroutine actually calculates the covariance at general nonzero values of both $\hat{\Delta R}$ and \hat{r} , although we only employ it for $\hat{\Delta R} = 0$ in this appendix; see lines 10 and 380. Also, for $\hat{\Delta R} = 0$, the parameter $Lg2 = (\Delta_L/\Delta_G)^2$ is not relevant and, hence, is entered as zero in line 380.

The exponential Gaussian forms for k_L and k_G are used in lines 1200 and 1210, while the triangular form for ρ is entered in line 1240. Any of these can be replaced, if desired, by forms more appropriate to the user.

The program is written in BASIC for the Hewlett Packard 9000 Computer Model 520. The designation DOUBLE denotes integer variables, not double precision. The output from the program is stored in data files AOT0, AOT1, AOT2, for $v = 0, 1, 2$, respectively.

```

10    Rc=0.                                ! DelR^
20    A=.2                                ! A(subA)
30    Gp=.001                             ! GAMMA'(subA)
40    Wlb2=1.                             ! (DelW(subL)/Beta)^2
50    Wgb2=25.                             ! (DelW(subG)/Beta)^2
60    Dtc=.01                             ! INCREMENT IN Tau^
70    Ntc=200                             ! NUMBER OF Tau^ VALUES
80    Tolerance=1.E-10
90    COM Af(0:40),C(0:80),Sq(0:80)
100   COM DOUBLE J                        ! INTEGER
110   DIM Kag(200),Tc(0:200),F0(0:200),F1(0:200),F2(0:200)
120   DOUBLE Ntc,K                        ! INTEGERS
130   FOR K=0 TO Ntc
140     Tc=K*Dtc                          ! Tau^
150     Rho=MAX(0.,1.-ABS(Tc))             ! Rho
160     T2=.5*Tc*Tc
170     K1=EXP(-Wlb2*T2)
180     Kg=EXP(-Wgb2*T2)
190     Kag(K)=(Rho*K1+Gp*Kg)/(1.+Gp)      ! INPUT COVARIANCE
200   NEXT K
210   A1=1./A                             ! A>0 REQUIRED
220   F0inf=PI
230   F1inf=FNFIinf(A,Gp)
240   F2inf=.25*PI*(1.+Gp)*(1.+Gp)
250   Af(0)=1.
260   FOR K=1 TO 40
270     J=K
280     Af(K)=T=Af(K-1)*A/K                ! A^K/K!
290     IF T<Tolerance THEN 320
300   NEXT K
310   PRINT "40 TERMS IN Af(*)"
320   FOR K=0 TO J*2
330     C(K)=T=K*A1+Gp
340     Sq(K)=1./SQRT(T)
350   NEXT K
360   FOR K=0 TO Ntc
370     Tc(K)=Tc=K*Dtc                     ! Tau^
380     CALL Myc(Rc,Tc,A,Gp,Wlb2,Wgb2,0.,F0(K),F1(K),F2(K))
390   NEXT K
400   MAT F0=F0-(F0inf)
410   MAT F1=F1-(F1inf)
420   MAT F2=F2-(F2inf)
430   MAT F0=F0/(F0(0))
440   MAT F1=F1/(F1(0))
450   MAT F2=F2/(F2(0))
460   PRINT "INFINITY: ";F0inf;F1inf;F2inf
470   PRINT "MINIMA: ";MIN(F0(*));MIN(F1(*));MIN(F2(*))
480   PRINT "AT Ntc: ";F0(Ntc);F1(Ntc);F2(Ntc)
490   CREATE DATA "AIT1",8
500   ASSIGN #1 TO "AIT1"
510   PRINT #1;Kag(*)
520   CREATE DATA "AOT0",8
530   ASSIGN #1 TO "AOT0"
540   PRINT #1;F0(*)
550   CREATE DATA "AOT1",8

```

```

560  ASSIGN #1 TO "AOT1"
570  PRINT #1;F1(*)
580  CREATE DATA "AOT2",8
590  ASSIGN #1 TO "AOT2"
600  PRINT #1;F2(*)
610  ASSIGN #1 TO *
620  Tcmax=Dtc*Ntc
630  GINIT 200/260
640  PLOTTER IS 505,"HPGL"
650  PRINTER IS 505
660  LIMIT PLOTTER 505,0,200,0,260
670  VIEWPORT 22,85,19,122
680  WINDOW 0.,1.,0.,1.
690  PRINT "VS5"
700  GRID .25,.25
710  PRINT "VS36"
720  PLOT Tc(*),Kag(*)
730  PENUP
740  PLOT Tc(*),F0(*)
750  PENUP
760  PLOT Tc(*),F1(*)
770  PENUP
780  PLOT Tc(*),F2(*)
790  PENUP
800  PAUSE
810  PRINTER IS CRT
820  PLOTTER 505 IS TERMINATED
830  END
840  !
850  DEF FNF1inf(A,Gp)                ! for v(=nu) = 1
860  Tol=1.E-18
870  Ag=A*Gp
880  T=1.
890  S=SQR(1.+Ag)
900  FOR M=2 TO 100
910  T=T*A/M
920  P=T*SQR(M+Ag)
930  S=S+P
940  IF P<S*Tol THEN 970
950  NEXT M
960  PRINT "100 TERMS IN FNF1inf"
970  T=Gp+A*S*S+2.*SQR(Ag)*S
980  RETURN EXP(-2.*A)*T
990  FNEND
1000 !
1010 SUB Bnu(Y,B0,B1,B2)              ! Bv(Y) for v=0,1,2
1020 IF Y>1. THEN PRINT "Y = 1 +";Y-1.
1030 IF Y>1. THEN Y=1.
1040 Y2=Y*Y
1050 Sq=SQR(1.-Y2)
1060 T=ASN(Y)+1.5707963267948966
1070 B0=T+T
1080 B1=Y*T+Sq
1090 B2=(.5+Y2)*T+1.5*Y*Sq
1100 SUBEND
1110 !

```

```

1120 SUB Mys(Rc,Tc,A,Gp,W1b2,Wgb2,Lg2,S0,S1,S2)
1130 COM Af(*),C(*),Sq(*)
1140 COM DOUBLE J ! INTEGER
1150 ALLOCATE Ap(0:J),Ap1(0:J)
1160 DOUBLE K,M1,M2,N,K1,K2 ! INTEGERS
1170 A1=1./A ! A>0 REQUIRED
1180 T2=.5*Tc*Tc
1190 R2=Rc*Rc
1200 K1=EXP(-R2-W1b2*T2)
1210 Kg=EXP(-Lg2*R2-Wgb2*T2)
1220 Ak=A1*K1
1230 Gk=Gp*Kg
1240 Rho=MAX(0.,1.-ABS(Tc)) ! Rho
1250 Rho1=1.-Rho
1260 Ap(0)=Ap1(0)=Pk=Pk1=1.
1270 FOR K=1 TO J
1280 Pk=Pk*Rho
1290 Pk1=Pk1*Rho1
1300 T=Af(K)
1310 Ap(K)=T*Pk
1320 Ap1(K)=T*Pk1
1330 NEXT K
1340 S0m1=S1m1=S2m1=0.
1350 FOR M1=0 TO J
1360 S0m2=S1m2=S2m2=0.
1370 FOR M2=0 TO J
1380 S0n=S1n=S2n=0.
1390 FOR N=0 TO J
1400 K1=N+M1
1410 K2=N+M2
1420 T=Ap(N)
1430 P=C(K1)*C(K2)
1440 Y=(N*Ak+Gk)*Sq(K1)*Sq(K2)
1450 CALL Bnu(Y,B0,B1,B2)
1460 S0n=S0n+T*B0
1470 S1n=S1n+T*SQR(P)*B1
1480 S2n=S2n+T*P*B2
1490 NEXT N
1500 T2=Ap1(M2)
1510 S0m2=S0m2+T2*S0n
1520 S1m2=S1m2+T2*S1n
1530 S2m2=S2m2+T2*S2n
1540 NEXT M2
1550 T1=Ap1(M1)
1560 S0m1=S0m1+T1*S0m2
1570 S1m1=S1m1+T1*S1m2
1580 S2m1=S2m1+T1*S2m2
1590 NEXT M1
1600 T=EXP(-A*(2.-Rho))
1610 S0=T*S0m1
1620 S1=T*S1m1
1630 S2=T*S2m1
1640 SUBEND

```

APPENDIX A.2 - EVALUATION OF COVARIANCE FUNCTION
FOR ZERO DELAY ($\hat{t}, \hat{t}' = 0$)

A program for the numerical evaluation of covariance $\hat{M}_y(\hat{\Delta R}, \hat{t})$ for $\hat{t}, \hat{t}' = 0$ is contained in this appendix. Inputs required of the user are A , Γ'_A , $(\Delta_L/\Delta_G)^2$, $\delta(\hat{\Delta R})$, $N(\hat{\Delta R})$, in lines 20 - 60. The tolerance for terminating the triple infinite sums is set at $1E-15$ in line 70. The output from the program is stored in data files AOR0, AOR1, AOR2, for $v = 0, 1, 2$, respectively. Other relevant comments are made in appendix A.1.

The limit of \hat{M}_y at $\hat{\Delta R} = \infty$ (when $\hat{t} = 0$) is given by the closed form results

$$\hat{M}_y(\infty, 0) = \begin{cases} \pi & \text{for } v = 0 \\ 1 + \Gamma'_A & \text{for } v = 1 \\ \frac{\pi}{4} \left[\frac{1}{\Delta} + (1 + \Gamma'_A)^2 \right] & \text{for } v = 2 \end{cases} \quad (\text{A.2-1})$$

These values have been subtracted from \hat{M}_y so that we can Bessel transform a function which tends to zero as $\hat{\Delta R} \rightarrow \infty$.

```

10   Tc=0.                ! Tau^
20   A=.2                 ! A(subA)
30   Gp=.001              ! GAMMA'(subA)
40   Lg2=5.               ! (De1L/De1G)^2
50   Drc=.005             ! INCREMENT IN De1R^
60   Nrc=900              ! NUMBER OF De1R^ VALUES
70   Tolerance=1.E-15
80   COM Af(0:40),C(0:80),Sq(0:80)
90   COM DOUBLE J         ! INTEGER
100  DIM Rc(0:900),Kag(0:900),F0(0:900),F1(0:900),F2(0:900)
110  DOUBLE Nrc,K         ! INTEGERS
120  A1=1./A              ! A>0 REQUIRED
130  F0inf=PI              ! LIMITS FOR
140  F1inf=1.+Gp           ! Rc=infinity
150  F2inf=.25*PI*((1.+Gp)*(1.+Gp)+A1) ! AND Tc=0
160  Af(0)=1.
170  FOR K=1 TO 40
180    J=K
190    Af(K)=T=Af(K-1)*A/K    ! A^K/K!
200    IF T<Tolerance THEN 230
210  NEXT K
220  PRINT "40 TERMS IN Af(*)"
230  FOR K=0 TO J*2
240    C(K)=T=K*A1+Gp
250    Sq(K)=1./SQRT(T)
260  NEXT K

```

```

270  FOR K=0 TO Nrc
280  Rc(K)=Rc=K*Drc          ! DeIR^
290  R2=Rc*Rc
300  K1=EXP(-R2)
310  Kg=EXP(-Lg2*R2)
320  Rho=MAX(0.,1.-ABS(Tc))  ! Rho
330  Kag(K)=(Rho*K1+Gp*Kg)/(1.+Gp) ! INPUT COVARIANCE
340  CALL Myc(Rc,Tc,A,Gp,0.,0.,Lg2,F0(K),F1(K),F2(K))
350  NEXT K
360  MAT F0=F0-(F0inf)
370  MAT F1=F1-(F1inf)
380  MAT F2=F2-(F2inf)
390  MAT F0=F0/(F0(0))
400  MAT F1=F1/(F1(0))
410  MAT F2=F2/(F2(0))
420  PRINT "INFINITY:";F0inf;F1inf;F2inf
430  PRINT "MINIMA: ";MIN(F0(*));MIN(F1(*));MIN(F2(*))
440  PRINT "AT Nrc: ";F0(Nrc);F1(Nrc);F2(Nrc)
450  CREATE DATA "AIR1",33
460  ASSIGN #1 TO "AIR1"
470  PRINT #1;Kag(*)
480  CREATE DATA "AOR0",33
490  ASSIGN #1 TO "AOR0"
500  PRINT #1;F0(*)
510  CREATE DATA "AOR1",33
520  ASSIGN #1 TO "AOR1"
530  PRINT #1;F1(*)
540  CREATE DATA "AOR2",33
550  ASSIGN #1 TO "AOR2"
560  PRINT #1;F2(*)
570  ASSIGN #1 TO *
580  Rcmax=Drc*Nrc
590  GINIT 200/260
600  PLOTTER IS 505,"HPGL"
610  PRINTER IS 505
620  LIMIT PLOTTER 505,0,200,0,260
630  VIEWPORT 22,85,19,122
640  WINDOW 0.,3.,0.,1.
650  PRINT "VS5"
660  GRID .5,.25
670  PRINT "VS36"
680  PLOT Rc(*),Kag(*)
690  PENUP
700  PLOT Rc(*),F0(*)
710  PENUP
720  PLOT Rc(*),F1(*)
730  PENUP
740  PLOT Rc(*),F2(*)
750  PENUP
760  PAUSE
770  PRINTER IS CRT
780  PLOTTER 505 IS TERMINATED
790  END
800  !
810  SUB Bnu(Y,B0,B1,B2)      ! Bu(Y) for v=0,1,2
820  ! SEE APPENDIX A.1
900  SUBEND
910  !
920  SUB Myc(Rc,Tc,A,Gp,W1b2,Wgb2,Lg2,S0,S1,S2)
930  ! SEE APPENDIX A.1
1440 SUBEND

```


APPENDIX A.3 - EVALUATION OF TEMPORAL INTENSITY
SPECTRUM FOR ZERO SEPARATION ($\hat{\Delta R} = 0$)

A program for the numerical evaluation of the Fourier transform of covariance $\hat{M}_y(0, \hat{r}) - \hat{M}_y(\infty)$ is contained in this appendix. Inputs required of the user are listed in lines 10 - 30. The data input, AOT0 or AOT1 or AOT2, as generated by means of the program in appendix A.1, is injected by means of lines 410, 600, and 790.

In order to keep the FFT (fast Fourier transform) size, N in lines 30 and 320, at reasonable values, the data sequence is collapsed, without any loss of accuracy, according to the method given in [8; pages 7 - 8] and [9; pages 13 - 16]. The integration rule documented here is the trapezoidal rule; this procedure is very accurate and efficient and is recommended for numerical Fourier transforms.

```

10      Ntc=200                      !  NUMBER OF Tau^ VALUES
20      Dtc=.01                     !  INCREMENT IN Tau^
30      N=1024                      !  SIZE OF FFT; N > Ntc REQUIRED
40      DOUBLE Ntc,N,N4,N2,Ns      !  INTEGERS
50      N4=N/4
60      N2=N/2
70      REDIM Cos(0:N4),X(0:N-1),Y(0:N-1)
80      DIM Cos(256),X(1023),Y(1023),A(200)
90      T=2.*PI/N
100     FOR Ns=0 TO N4
110     Cos(Ns)=COS(T*Ns)          !  QUARTER-COSINE TABLE IN Cos(*)
120     NEXT Ns
130     GINIT 200/260
140     PLOTTER IS 505,"HPGL"
150     PRINTER IS 505
160     LIMIT PLOTTER 505,0,200,0,260
170     VIEWPORT 22,85,19,122
180     WINDOW 0,N2,-5,1
190     PRINT "VS5"
200     GRID N/10,1
210     PRINT "VS36"
220     ASSIGN #1 TO "AIT1"
230     READ #1;A(*)
240     MAT X=(0.)
250     MAT Y=(0.)
260     X(0)=.5*A(0)
270     FOR Ns=1 TO Ntc-1
280     X(Ns)=A(Ns)
290     NEXT Ns
300     X(Ntc)=.5*A(Ntc)

```

```

310   MAT X=X*(Dtc*4.)
320   CALL Fft14(N,Cos(*),Y(*),Y(*))
330   FOR Ns=0 TO N2
340   Ar=X(Ns)
350   IF Ar>0. THEN 380
360   PENUP
370   GOTO 390
380   PLOT Ns,LGT(Ar)
390   NEXT Ns
400   PENUP
410   ASSIGN #1 TO "AOT0"
420   READ #1;A(*)
430   MAT X=(0.)
440   MAT Y=(0.)
450   X(0)=.5*A(0)
460   FOR Ns=1 TO Ntc-1
470   X(Ns)=A(Ns)
480   NEXT Ns
490   X(Ntc)=.5*A(Ntc)
500   MAT X=X*(Dtc*4.)
510   CALL Fft14(N,Cos(*),X(*),Y(*))
520   FOR Ns=0 TO N2
530   Ar=X(Ns)
540   IF Ar>0. THEN 570
550   PENUP
560   GOTO 580
570   PLOT Ns,LGT(Ar)
580   NEXT Ns
590   PENUP
600   ASSIGN #1 TO "AOT1"
610   READ #1;A(*)
620   MAT X=(0.)
630   MAT Y=(0.)
640   X(0)=.5*A(0)
650   FOR Ns=1 TO Ntc-1
660   X(Ns)=A(Ns)
670   NEXT Ns
680   X(Ntc)=.5*A(Ntc)
690   MAT X=X*(Dtc*4.)
700   CALL Fft14(N,Cos(*),X(*),Y(*))
710   FOR Ns=0 TO N2
720   Ar=X(Ns)
730   IF Ar>0. THEN 760
740   PENUP
750   GOTO 770
760   PLOT Ns,LGT(Ar)
770   NEXT Ns
780   PENUP
790   ASSIGN #1 TO "AOT2"
800   READ #1;A(*)
810   MAT A=A/(A(0))
820   MAT X=(0.)
830   MAT Y=(0.)

```

```

840      X(0)=.5*A(0)
850      FOR Ns=1 TO Ntc-1
860      X(Ns)=A(Ns)
870      NEXT Ns
880      X(Ntc)=.5*A(Ntc)
890      MAT X=X*(Dtc*4.)
900      CALL Fft14(N,Cos(*),X(*),Y(*))
910      FOR Ns=0 TO N2
920      Ar=X(Ns)
930      IF Ar>0. THEN 960
940      PENUP
950      GOTO 970
960      PLOT Ns,LGT(Ar)
970      NEXT Ns
980      PENUP
990      PAUSE
1000     END
1010     !
1020     SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
1030     DOUBLE Log2n,N1,N2,N3,N4,J,K ! INTEGERS < 2^31 = 2,147,483,648
1040     DOUBLE I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,L(0:13)
1050     IF N=1 THEN SUBEXIT
1060     IF N>2 THEN 1140
1070     A=X(0)+X(1)
1080     X(1)=X(0)-X(1)
1090     X(0)=A
1100     A=Y(0)+Y(1)
1110     Y(1)=Y(0)-Y(1)
1120     Y(0)=A
1130     SUBEXIT
1140     A=LOG(N)/LOG(2.)
1150     Log2n=A
1160     IF ABS(A-Log2n)<1.E-8 THEN 1190
1170     PRINT "N =";N;"IS NOT A POWER OF 2; DISALLOWED."
1180     PAUSE
1190     N1=N/4
1200     N2=N1+1
1210     N3=N2+1
1220     N4=N3+N1
1230     FOR I1=1 TO Log2n
1240     I2=2^(Log2n-I1)
1250     I3=2*I2
1260     I4=N/I3
1270     FOR I5=1 TO I2
1280     I6=(I5-1)*I4+1
1290     IF I6<=N2 THEN 1330
1300     A1=-Cos(N4-I6-1)
1310     A2=-Cos(I6-N1-1)
1320     GOTO 1350
1330     A1=Cos(I6-1)
1340     A2=-Cos(N3-I6-1)
1350     FOR I7=0 TO N-I3 STEP I3
1360     I8=I7+I5-1
1370     I9=I8+I2
1380     T1=X(I8)
1390     T2=X(I9)

```

```

1400   T3=Y(I8)
1410   T4=Y(I9)
1420   A3=T1-T2
1430   A4=T3-T4
1440   X(I8)=T1+T2
1450   Y(I8)=T3+T4
1460   X(I9)=A1*A3-A2*A4
1470   Y(I9)=A1*A4+A2*A3
1480   NEXT I7
1490   NEXT I5
1500   NEXT I1
1510   I1=Log2n+1
1520   FOR I2=1 TO 14
1530     L(I2-1)=1
1540     IF I2>Log2n THEN 1560
1550     L(I2-1)=2^(I1-I2)
1560   NEXT I2
1570   K=0
1580   FOR I1=1 TO L(13)
1590     FOR I2=I1 TO L(12) STEP L(13)
1600       FOR I3=I2 TO L(11) STEP L(12)
1610         FOR I4=I3 TO L(10) STEP L(11)
1620           FOR I5=I4 TO L(9) STEP L(10)
1630             FOR I6=I5 TO L(8) STEP L(9)
1640               FOR I7=I6 TO L(7) STEP L(8)
1650                 FOR I8=I7 TO L(6) STEP L(7)
1660                   FOR I9=I8 TO L(5) STEP L(6)
1670                     FOR I10=I9 TO L(4) STEP L(5)
1680                       FOR I11=I10 TO L(3) STEP L(4)
1690                         FOR I12=I11 TO L(2) STEP L(3)
1700                           FOR I13=I12 TO L(1) STEP L(2)
1710                             FOR I14=I13 TO L(0) STEP L(1)
1720                               J=I14-1
1730                               IF K>J THEN 1800
1740                               A=X(K)
1750                               X(K)=X(J)
1760                               X(J)=A
1770                               A=Y(K)
1780                               Y(K)=Y(J)
1790                               Y(J)=A
1800                               K=K+1
1810                             NEXT I14
1820                           NEXT I13
1830                         NEXT I12
1840                       NEXT I11
1850                     NEXT I10
1860                   NEXT I9
1870                 NEXT I8
1880               NEXT I7
1890             NEXT I6
1900           NEXT I5
1910         NEXT I4
1920       NEXT I3
1930     NEXT I2
1940   NEXT I1
1950   SUBEND

```

APPENDIX A.4 - EVALUATION OF WAVENUMBER INTENSITY SPECTRUM FOR ZERO DELAY ($\hat{r}, \hat{r}' = 0$)

A program for the numerical evaluation of the zeroth-order Bessel transform of covariance $\hat{M}_y(\hat{\Delta R}, 0) - \hat{M}_y(\infty)$ is contained in this appendix. Inputs required of the user are listed in lines 10 - 40 and are coupled to appendix A.2, where the data input, AOR0 or AOR1 or AOR2, was generated. The numerical Bessel transform is accomplished by means of Simpson's rule with end correction [11; pages 414 - 418], and is exceedingly accurate for the small increment, .005, in $\hat{\Delta R}$ employed in line 30.

```

10   Dkc=.4                ! INCREMENT IN k^
20   Nkc=200              ! NUMBER OF k^ VALUES
30   Drc=.005             ! INCREMENT IN DeIR^
40   Nrc=900              ! NUMBER OF DeIR^ VALUES
50   DOUBLE Nrc,Nkc,I,Ns  ! INTEGERS
60   REDIM C(0:Nrc)
70   REDIM Wi(0:Nkc),W0(0:Nkc),W1(0:Nkc),W2(0:Nkc)
80   DIM C(900),Wi(200),W0(200),W1(200),W2(200)
90   ASSIGN #1 TO "AIR1"
100  READ #1;C(*)
110  FOR I=0 TO Nkc
120    Kc=I*Dkc            ! k^
130    T=Kc*Drc
140    Se=So=0.
150    FOR Ns=1 TO Nrc-1 STEP 2
160      So=So+Ns*FNJo(T*Ns)*C(Ns)
170    NEXT Ns
180    FOR Ns=2 TO Nrc-2 STEP 2
190      Se=Se+Ns*FNJo(T*Ns)*C(Ns)
200    NEXT Ns
210    Wi(I)=C(0)+16.*So+14.*Se
220  NEXT I
230  MAT Wi=Wi*(Drc*Drc*2.*PI/15.)
240  ASSIGN #1 TO "AOR0"
250  READ #1;C(*)
260  FOR I=0 TO Nkc
270    Kc=I*Dkc
280    T=Kc*Drc
290    Se=So=0.
300    FOR Ns=1 TO Nrc-1 STEP 2
310      So=So+Ns*FNJo(T*Ns)*C(Ns)
320    NEXT Ns
330    FOR Ns=2 TO Nrc-2 STEP 2
340      Se=Se+Ns*FNJo(T*Ns)*C(Ns)
350    NEXT Ns
360    W0(I)=C(0)+16.*So+14.*Se
370  NEXT I
380  MAT W0=W0*(Drc*Drc*2.*PI/15.)

```

```

390  ASSIGN #1 TO "AOR1"
400  READ #1;C(*)
410  FOR I=0 TO Nkc
420    Kc=I*Dkc
430    T=Kc*Drc
440    Se=So=0.
450    FOR Ns=1 TO Nrc-1 STEP 2
460      So=So+Ns*FNJo(T*Ns)*C(Ns)
470    NEXT Ns
480    FOR Ns=2 TO Nrc-2 STEP 2
490      Se=Se+Ns*FNJo(T*Ns)*C(Ns)
500    NEXT Ns
510    W1(I)=C(0)+16.*So+14.*Se
520  NEXT I
530  MAT W1=W1*(Drc*Drc*2.*PI/15.)
540  ASSIGN #1 TO "AOR2"
550  READ #1;C(*)
560  ASSIGN #1 TO *
570  FOR I=0 TO Nkc
580    Kc=I*Dkc
590    T=Kc*Drc
600    Se=So=0.
610    FOR Ns=1 TO Nrc-1 STEP 2
620      So=So+Ns*FNJo(T*Ns)*C(Ns)
630    NEXT Ns
640    FOR Ns=2 TO Nrc-2 STEP 2
650      Se=Se+Ns*FNJo(T*Ns)*C(Ns)
660    NEXT Ns
670    W2(I)=C(0)+16.*So+14.*Se
680  NEXT I
690  MAT W2=W2*(Drc*Drc*2.*PI/15.)
700  GINIT 200/260
710  PLOTTER IS 505,"HPGL"
720  PRINTER IS 505
730  LIMIT PLOTTER 505,0,200,0,260
740  VIEWPORT 22,85,19,122
750  WINDOW 0,Nkc,-9,1
760    PRINT "VS5"
770  GRID 25,1
780    PRINT "VS36"
790  FOR I=0 TO Nkc
800    W=Wi(I)
810    IF W>0. THEN 840
820  PENUP
830  GOTO 850
840  PLOT I,LGT(W)
850  NEXT I
860  PENUP

```

```

870   FOR I=0 TO Hkc
880   W=W0(I)
890   IF W>0. THEN 920
900   PENUP
910   GOTO 930
920   PLOT I,LGT(W)
930   NEXT I
940   PENUP
950   FOR I=0 TO Hkc
960   W=W1(I)
970   IF W>0. THEN 1000
980   PENUP
990   GOTO 1010
1000  PLOT I,LGT(W)
1010  NEXT I
1020  PENUP
1030  FOR I=0 TO Hkc
1040  W=W2(I)
1050  IF W>0. THEN 1080
1060  PENUP
1070  GOTO 1090
1080  PLOT I,LGT(W)
1090  NEXT I
1100  PENUP
1110  PAUSE
1120  PRINTER IS CRT
1130  PLOTTER 505 IS TERMINATED
1140  END
1150  !
1160  DEF FNJo(X)      ! Jo(X) FOR ALL X
1170  Y=ABS(X)
1180  IF Y>8. THEN 1280
1190  T=Y*Y            ! HART, #5845
1200  P=2271490439.5536033-T*(5513584.5647707522-T*5292.6171303845574)
1210  P=2334489171877869.7-T*(47765559442673.588-T*(462172225031.71803-T*P))
1220  P=185962317621897804.-T*(44145829391815982.-T*P)
1230  Q=204251483.52134357+T*(494030.79491813972+T*(884.72036756175504+T))
1240  Q=2344750013658996.8+T*(15015462449769.752+T*(64398674535.133256+T*Q))
1250  Q=185962317621897733.+T*Q
1260  Jo=P/Q
1270  RETURN Jo
1280  Z=8./Y          ! HART, #6546 & 6946
1290  T=Z*Z
1300  Pn=2204.5010439651804+T*(128.67758574871419+T*.90047934748028803)
1310  Pn=8554.8225415066617+T*(8894.4375329606194+T*Pn)
1320  Pd=2214.0488519147104+T*(130.88490049992388+T)
1330  Pd=8554.8225415066628+T*(8903.8361417095954+T*Pd)
1340  Qn=13.990976865960680+T*(1.0497327982345548+T*.00935259532940319)
1350  Qn=-37.510534954957112-T*(46.093826814625175+T*Qn)
1360  Qd=921.56697552653090+T*(74.428389741411179+T)
1370  Qd=2400.6742371172675+T*(2971.9837452084920+T*Qd)
1380  T=Y-.78539816339744828
1390  Jo=.28209479177387820*SQR(Z)*(COS(T)*Pn/Pd-SIN(T)*Z*Qn/Qd)
1400  RETURN Jo
1410  FEND

```

APPENDIX A.5 - EVALUATION OF PHASE MODULATION INTENSITY SPECTRUM

The normalized covariance function for phase modulation is given by (2.50) in the main text, namely

$$k_o(\zeta) = \exp\left[-\Gamma'_A \mu_P^2 [1 - \exp(-\zeta)] - A[2 - \rho(\zeta)] + \right. \\ \left. + 2A [1 - \rho(\zeta)] \exp\left(-\mu_P^2/A\right) + A \rho(\zeta) \exp\left(-\frac{\mu_P^2}{A} [1 - \exp(-b\zeta)]\right)\right] \quad (\text{A.5-1})$$

for $\zeta \geq 0$, where ζ is the time delay and $\rho(\zeta)$ is the temporal normalized covariance of the field process. Also $\mu_P^2 = \mu_{PG}^2$. Since (A.5-1) involves an exponential of an exponential of an exponential, and because a wide range of parameter values are of interest, care must be taken in numerical evaluation of this covariance and its transform.

Observe first that

$$k_o(0) = 1 \quad \text{since } \rho(0) = 1. \quad (\text{A.5-2})$$

Also, as delay $\zeta \rightarrow +\infty$, then $\rho \rightarrow 0$, giving

$$k_o(\infty) = \exp\left[-\Gamma'_A \mu_P^2 - 2A + 2A \exp\left(-\mu_P^2/A\right)\right] \neq 0. \quad (\text{A.5-3})$$

The spectrum of interest is given by

$$W_o(\omega) = 4 \int_0^{\infty} d\zeta \cos(\omega\zeta) k_o(\zeta) \quad \text{for } \omega \geq 0; \quad \omega = 2\pi f. \quad (\text{A.5-4})$$

The nonzero value of (A.5-3) at $\zeta = \infty$ leads to an impulse in spectrum $W_o(\omega)$ at $\omega = 0$. This limiting value, $k_o(\infty)$, must be subtracted from covariance (A.5-1) prior to the numerical Fourier transform indicated by (A.5-4).

For $\Gamma'_A \mu_P^2 \ll 1$, the term

$$\exp\left\{-\Gamma'_A \mu_P^2 [1 - \exp(-\zeta)]\right\} \quad (\text{A.5-5})$$

approaches its limiting value at $\zeta = +\infty$ as follows:

$$\begin{aligned} & \exp\left\{-\Gamma'_A \mu_P^2 [1 - \exp(-\zeta)]\right\} - \exp\left\{-\Gamma'_A \mu_P^2\right\} = \\ & = \exp\left\{-\Gamma'_A \mu_P^2\right\} \left[\exp\left\{\Gamma'_A \mu_P^2 \exp(-\zeta)\right\} - 1\right] = \\ & \sim \exp\left\{-\Gamma'_A \mu_P^2\right\} \Gamma'_A \mu_P^2 \exp(-\zeta) . \end{aligned} \quad (\text{A.5-6})$$

This is a fairly rapid decay with ζ and will not lead to numerical difficulty when $\Gamma'_A \mu_P^2 \ll 1$.

For large $b\mu_P^2/A$, the term

$$\exp\left\{-\frac{\mu_P^2}{A} [1 - \exp(-b\zeta)]\right\} \quad (\text{A.5-7})$$

is very sharp near $\zeta = 0$; in fact, it is given approximately by

$$\exp\left\{-\frac{\mu_P^2}{A} b\zeta\right\} \quad \text{for } \zeta \text{ near } 0 . \quad (\text{A.5-8})$$

Therefore, we define the sharp component of covariance $k_o(\zeta)$ as

$$k_s(\zeta) = \exp\left[-A + A \exp\left\{-\frac{\mu_P^2}{A} b\zeta\right\}\right] - \exp(-A) \quad \text{for all } \zeta . \quad (\text{A.5-9})$$

Then

$$k_s(0) = 1 - \exp(-A) , \quad k_s(\infty) = 0 . \quad (\text{A.5-10})$$

Now we let

$$\begin{aligned} k_o(\zeta) &= [k_o(\zeta) - k_s(\zeta)] + k_s(\zeta) = \\ &\equiv k_f(\zeta) + k_s(\zeta) , \end{aligned} \quad (\text{A.5-11})$$

where $k_f(\zeta)$ is a flat function near $\zeta = 0$. Then we can express the desired difference as

$$\begin{aligned} k_o(\zeta) - k_o(\infty) &= [k_f(\zeta) - k_o(\infty)] + k_s(\zeta) = \\ &\equiv k_1(\zeta) + k_s(\zeta) , \end{aligned} \quad (\text{A.5-12})$$

where functions $k_1(\zeta)$ and $k_s(\zeta)$ both decay to 0 at $\zeta = \infty$. We now employ two separate FFTs on each of the functions in (A.5-12). The sharp component, $k_s(\zeta)$, must be sampled with a very small increment, $\Delta\zeta$, when $b\mu_p^2/A$ is large. On the other hand, the flat component

$$k_1(\zeta) = k_f(\zeta) - k_o(\infty) \quad (\text{A.5-13})$$

can be sampled in a coarser fashion. Finally, if $b\mu_p^2/A$ is moderate, we work directly with $k_o(\zeta) - k_o(\infty)$ without breaking it into any components.

Two programs are furnished in this appendix, one for moderate $b\mu_p^2/A$, and the other for the flat component (A.5-13) when $b\mu_p^2/A$ is large. For sake of brevity, the Fourier transform of the sharp component (A.5-9) is straightforward and is not presented. The particular covariance $\rho(\zeta)$ adopted is triangular,

$$\rho(\zeta) = 1 - \frac{|\zeta|}{\zeta_c} \quad \text{for } |\zeta| < \zeta_c , \quad 0 \text{ otherwise} , \quad (\text{A.5-14})$$

but can easily be replaced. The parameter ζ_c is the cutoff value of covariance $\rho(\zeta)$.

The number of samples, N , taken of the covariance, in order to perform the FFT of (A.5-4), is rather large, so as to guarantee a very small value of truncation error at the upper end of the integral, despite the small increment $\Delta\zeta$. In order to keep the FFT size, M_f , at reasonable values, the data sequence is collapsed without any loss of accuracy according to the method given in [8; pages 7 - 8] and [9; pages 13 - 16]. The trapezoidal rule is used to approximate the integral in (A.5-4), for reasons given in [8; appendix A].

```

10  ! SPECTRUM FOR PHASE MODULATION - MODERATE
20  Mup=1.                ! MUsupP
30  Gp=.001               ! Gamma'
40  Bs=1.                 ! b
50  A=.2                  ! A
60  Zc=2.*PI              ! Rho(Z) = 0 for |Z|>Zc; Z=zeta
70  Delz=.005             ! Zeta increment
80  N=60000               ! Maximum number of samples of ko(zeta)
90  Mf=16384              ! Size of FFT
100 DOUBLE N,Mf,Ms,Ns     ! INTEGERS
110 DIM X(16384),Y(16384),Cos(4096)
120 REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130 MAT X=(0.)
140 MAT Y=(0.)
150 T=2.*PI/Mf
160 FOR Ms=0 TO Mf/4
170   Cos(Ms)=COS(T*Ms)    ! QUARTER-COSINE TABLE
180 NEXT Ms
190 Ta=Gp*Mup*Mup
200 IF A=0. THEN 220
210 Tb=Mup*Mup/A
220 Tc=2.*A*FNExp(Tb)
230 Kinf=FNExp(Ta+2.*A-Tc) ! CORRELATION AT INFINITY
240 COM A,Bs,Zc,Ta,Tb,Tc,Kinf
250 T=1.-Kinf
260   PRINT 0,T
270   X(0)=T*.5            ! TRAPEZOIDAL RULE
280   FOR Ns=1 TO N
290     Corr=FNKo(Ns*Delz) ! CORRELATION ko(zeta)
300     IF Ns<6 THEN PRINT Ns,Corr
310     IF ABS(Corr)<1.E-30 THEN 350
320     Ms=Ns MODULO Mf    ! COLLAPSING
330     X(Ms)=X(Ms)+Corr
340   NEXT Ns
350   PRINT "Final value of Corr =";Corr;"  Ns =";Ns
360   MAT X=X*(Delz*4.)
370   CALL Fft14(Mf,Cos(*),X(*),Y(*))

```

```

380  GINIT
390  PLOTTER IS "GRAPHICS"
400  GRAPHICS ON
410  WINDOW -2,2,-60,0
420  LINE TYPE 3
430  GRID 1,10
440  LINE TYPE 1
450  Delf=1./(Mf*Delz)
460  FOR Ms=1 TO Mf/2
470  F=Ms*Delf          ! FREQUENCY
480  PLOT LGT(F),10.*LGT(X(Ms))
490  NEXT Ms
500  PENUP
510  PAUSE
520  END
530  !
540  DEF FNExp(Xminus)    ! EXP(-X) WITHOUT UNDERFLOW
550  IF Xminus>708.3 THEN RETURN 0.
560  RETURN EXP(-Xminus)
570  FNEED
580  !
590  DEF FNKo(Zeta)       ! CORRELATION ko(zeta)
600  COM A,Bs,Zc,Ta,Tb,Tc,Kinf
610  Rho=MAX(0.,1.-Zeta/Zc) ! TRIANGULAR RHO
620  E1=Ta*(1.-FNExp(Zeta))
630  E2=Tb*(1.-FNExp(Bs*Zeta))
640  E3=A*Rho*FNExp(E2)
650  RETURN FNExp(E1+A*(2.-Rho)-Tc*(1.-Rho)-E3)-Kinf
660  FNEED
670  !
680  SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
690  ! SEE APPENDIX A.3

```

```

10  ! SPECTRUM FOR PHASE MODULATION - FLAT COMPONENT
20  Mup=1.          ! MUsupP
30  Gp=.001        ! Gamma'
40  Bs=1.          ! b
50  A=0.           ! A
60  Zc=2.*PI       ! Rho(Z) = 0 for |Z|>Zc; Z=zeta
70  Delz=.005      ! Zeta increment
80  N=60000        ! Maximum number of samples of k1(zeta)
90  Mf=16384       ! Size of FFT
100  DOUBLE N,Mf,Ms,Ns ! INTEGERS
110  DIM X(16384),Y(16384),Cos(4096)
120  REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130  MAT X=(0.)
140  MAT Y=(0.)
150  T=2.*PI/Mf
160  FOR Ms=0 TO Mf/4
170  Cos(Ms)=COS(T*Ms) ! QUARTER-COSINE TABLE
180  NEXT Ms
190  Ta=Gp*Mup*Mup
200  IF A=0. THEN 220

```

```

210 Tb=Mup*Mup/A
220 Tc=2.*A*FNExp(Tb)
230 Tb=5.E55
240 Kinf=FNExp(Ta+2.*A-Tc) ! CORRELATION AT INFINITY
250 Ea=FNExp(A)
260 Tbb=Tb*Bb
270 COM A,Bs,Zc,Ta,Tb,Tc,Kinf,Ea,Tbb
280 T=1.-Kinf-(1.-Ea) ! SUBTRACT SHARP COMPONENT
290 PRINT 0,T
300 X(0)=T*.5 ! TRAPEZOIDAL RULE
310 FOR Ns=1 TO N
320 Corr=FNK1(Ns*Delz) ! CORRELATION k1(zeta)
330 IF Ns<6 THEN PRINT Ns,Corr
340 IF ABS(Corr)<1.E-30 THEN 380
350 Ms=Ns MODULO Mf ! COLLAPSING
360 X(Ms)=X(Ms)+Corr
370 NEXT Ns
380 PRINT "Final value of Corr =";Corr;" Ns =";Ns
390 MAT X=X*(Delz*4.)
400 CALL Fft14(Mf,Cos(*),X(*),Y(*))
410 GINIT
420 PLOTTER IS "GRAPHICS"
430 GRAPHICS ON
440 WINDOW -2,2,-60,0
450 LINE TYPE 3
460 GRID 1,10
470 LINE TYPE 1
480 Delf=1./(Mf*Delz)
490 FOR Ms=1 TO Mf/2
500 F=Ms*Delf ! FREQUENCY
510 T=X(Ms)
520 IF T>0. THEN 550
530 PENUP
540 GOTO 560
550 PLOT LGT(F),10.*LGT(T)
560 NEXT Ms
570 PENUP
580 PAUSE
590 END
600 !
610 DEF FNExp(Xminus) ! EXP(-X) WITHOUT UNDERFLOW
620 IF Xminus>708.3 THEN RETURN 0.
630 RETURN EXP(-Xminus)
640 FNEED
650 !
660 DEF FNK1(Zeta) ! CORRELATION k1(zeta)
670 COM A,Bs,Zc,Ta,Tb,Tc,Kinf,Ea,Tbb
680 Rho=MAX(0.,1.-Zeta/Zc) ! TRIANGULAR RHO
690 E1=Ta*(1.-FNExp(Zeta))
700 E2=Tb*(1.-FNExp(Bs*Zeta))
710 E3=A*Rho*FNExp(E2)
720 E4=FNExp(Tbb*Zeta)
730 Sharp=FNExp(A*(1.-E4))-Ea ! ks(zeta)
740 RETURN FNExp(E1+A*(2.-Rho)-Tc*(1.-Rho)-E3)-Kinf-Sharp
750 FNEED
760 !
770 SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
780 ! SEE APPENDIX A.3

```

APPENDIX A.6 - EVALUATION OF FREQUENCY MODULATION INTENSITY SPECTRUM

The normalized covariance function for frequency modulation is given by (2.48) in the main text, namely

$$k_0(\zeta) = \exp\left[-\Gamma'_A \mu_F^2 [\exp(-\zeta) + \zeta - 1] - A[2 - \rho(\zeta)] + A \rho(\zeta) \exp\left(-\frac{\mu_F^2}{Ab^2} [\exp(-b\zeta) + b\zeta - 1]\right)\right] \quad \text{for } \zeta \geq 0, \quad (\text{A.6-1})$$

where ζ is the time delay and $\rho(\zeta)$ is the temporal normalized covariance of the field process. Also, $\mu_F^2 = \mu_{FG}^2$ and $b = \Delta\omega_A/\Delta\omega_N$. Since (A.6-1) involves an exponential of an exponential of an exponential, and because a wide range of parameter values are of interest, care must be taken in numerical evaluation of this covariance and its transform.

Observe that

$$k_0(0) = 1, \quad \text{because } \rho(0) = 1. \quad (\text{A.6-2})$$

Also, as delay $\zeta \rightarrow +\infty$, then $\rho \rightarrow 0$, giving

$$k_0(\zeta) \sim \exp\left[-\Gamma'_A \mu_F^2 (\zeta - 1) - 2A\right] \equiv k_1(\zeta) \quad \text{for } \zeta > 0. \quad (\text{A.6-3})$$

This term, $k_1(\zeta)$, decays slowly with ζ if $\Gamma'_A \mu_F^2 \ll 1$.

The spectrum of interest is given by

$$W_0(\omega) = 4 \int_0^{\infty} d\zeta \cos(\omega\zeta) k_0(\zeta) \quad \text{for } \omega \geq 0; \quad \omega = 2\pi f. \quad (\text{A.6-4})$$

The spectrum corresponding to the limiting component, $k_1(\zeta)$ in (A.6-3), is directly available in closed form as

$$\begin{aligned}
 W_1(\omega) &= 4 \int_0^{\infty} d\zeta \cos(\omega\zeta) k_1(\zeta) = \\
 &= \exp\left(\Gamma'_A \mu_F^2 - 2A\right) \frac{4 \Gamma'_A \mu_F^2}{\left(\Gamma'_A \mu_F^2\right)^2 + \omega^2} . \quad (A.6-5)
 \end{aligned}$$

If $\Gamma'_A \mu_F^2 \ll 1$, this latter quantity is large and very sharply peaked at $\omega = 0$; hence, this term has been subtracted out and handled separately when $\Gamma'_A \mu_F^2 \ll 1$. The residual covariance, $k_0(\zeta) - k_1(\zeta)$, then decays very rapidly with ζ and is easily handled directly by means of an FFT. This breakdown is not necessary when $\Gamma'_A \mu_F^2 \sim 1$ and is avoided, then, by handling $k_0(\zeta)$ directly in one FFT.

For $\mu_F^2/A \gg 1$, the term

$$\exp\left(-\frac{\mu_F^2}{Ab^2} [\exp(-b\zeta) + b\zeta - 1]\right) \quad (A.6-6)$$

inside the exponential in (A.6-1) behaves like

$$\exp\left(-\frac{\mu_F^2}{A} \frac{1}{2} \zeta^2\right) \quad \text{near } \zeta = 0 , \quad (A.6-7)$$

where its major sharp contribution arises. For example, if $\mu_F = 50$, $A = 1$, then increment $\Delta\zeta = .005$ leads to values for (A.6-7) of

$$\exp(-0.156 n^2) \text{ at } \zeta = n \Delta\zeta , \quad (A.6-8)$$

which is adequately sampled in order to track its dominant contribution; the actual sequence of values is 1, .856, .536, .247, .083. For smaller μ_F^2/A , this sampling interval is more than adequate.

Two programs are furnished in this appendix, one each for the cases of large and small $\Gamma'_A \mu_F^2$. The particular covariance $\rho(\zeta)$ adopted is triangular,

$$\rho(\zeta) = 1 - \frac{|\zeta|}{\zeta_c} \quad \text{for } |\zeta| < \zeta_c, \quad 0 \text{ otherwise,} \quad (\text{A.6-9})$$

but can easily be replaced. The parameter ζ_c is the cutoff covariance value and is specified numerically in the examples in figures 3.9a through 3.10b.

```

10  ! SPECTRUM FOR FREQUENCY MODULATION - LARGE  $G_p \mu_F^2$ 
20  Muf=50.                ! MU=ubF
30  Gp=.001                ! Gamma'
40  Bs=1.                  ! b
50  A=.2                   ! A
60  Zc=2.*PI               ! Rho(Z) = 0 for |Z|>Zc; Z=zeta
70  Delz=.005              ! Zeta increment
80  N=60000                ! Maximum number of samples of ko(zeta)
90  Mf=16384               ! Size of FFT
100 DOUBLE N,Mf,Ms,Ns      ! INTEGERS
110 DIM X(16384),Y(16384),Cos(4096)
120 REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130 MAT X=(0.)
140 MAT Y=(0.)
150 T=2.*PI/Mf
160 FOR Ms=0 TO Mf/4
170  Cos(Ms)=COS(T*Ms)      ! QUARTER-COSINE TABLE
180 NEXT Ms
190 Ta=Gp*Muf*Muf
200 IF A=0. THEN 220
210 Tb=Muf*Muf/(A*Bs*Bs)
220 Tc=FNExp(2.*A-Ta)*Ta
230 Td=Ta*Ta
240 COM A,Bs,Zc,Ta,Tb
250 X(0)=.5                ! TRAPEZOIDAL RULE
260 FOR Ns=1 TO N
270  Corr=FNKo(Ns*Delz)     ! CORRELATION ko(zeta)
280  IF Corr<1.E-20 THEN 320
290  Ms=Ns MODULO Mf        ! COLLAPSING
300  X(Ms)=X(Ms)+Corr
310 NEXT Ns
320 PRINT "Final value of Corr =";Corr;"  Ns =";Ns
330 MAT X=X*(Delz)
340 CALL Fft14(Mf,Cos(*),X(*),Y(*))

```



```

350  GINIT
360  PLOTTER IS "GRAPHICS"
370  GRAPHICS ON
380  WINDOW -4,2,-70,30
390  LINE TYPE 3
400  GRID 1,10
410  LINE TYPE 1
420  Delf=1./(Mf*Delz)
430  FOR Ms=1 TO Mf/2
440  F=Ms*Delf ! FREQUENCY
450  T=X(Ms)
460  IF T>0. THEN 490
470  PENUP
480  GOTO 500
490  PLOT LGT(F),10.*LGT(T)
500  NEXT Ms
510  PENUP
520  Add=X(0)-Tc/Td ! ORIGIN CORRECTION
530  F=1.E-4
540  FOR Ns=1 TO 100
550  W=2.*PI*F
560  W1=Tc/(Td+W*W)
570  T=W1+Add
580  IF T>0. THEN 610
590  PENUP
600  GOTO 620
610  PLOT LGT(F),10.*LGT(W1+Add)
620  F=F*1.1 ! FREQUENCY
630  IF F>Delf THEN 650
640  NEXT Ns
650  PENUP
660  PAUSE
670  END
680  !
690  DEF FNExp(Xminus) ! EXP(-X) WITHOUT UNDERFLOW
700  IF Xminus>708.3 THEN RETURN 0.
710  RETURN EXP(-Xminus)
720  FNEND
730  !
740  DEF FNKo(Zeta) ! CORRELATION ko(zeta)
750  COM A,Bs,Zc,Ta,Tb
760  E1=FNExp(Zeta)+Zeta-1.
770  T=Bs*Zeta
780  E2=FNExp(T)+T-1.
790  Rho=MAX(0.,1-Zeta/Zc) ! TRIANGULAR RHO
800  T=Ta*E1+A*(2.-Rho)-A*Rho*FNExp(Tb*E2)
810  RETURN FNExp(T)
820  FNEND
830  !
840  SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! NK=2^14=16384; 0 SUBS
850  ! SEE APPENDIX A.3

```

```

10  ! SPECTRUM FOR FREQUENCY MODULATION - SMALL  $G_p * Muf^2$ 
20  Muf=1.                ! MUsubF
30  Gp=.001               ! Gamma'
40  Bs=1.                 ! b
50  A=.2                  ! A
60  Zc=2.*PI              !  $\text{Rho}(Z) = 0$  for  $|Z| > Zc$ ;  $Z = \text{zeta}$ 
70  Delz=.005             ! Zeta increment
80  N=10000               ! Maximum number of samples of  $k_0(\text{zeta})$ 
90  Mf=8192               ! Size of FFT
100 DOUBLE N,Mf,Ms,Ns     ! INTEGERS
110 DIM X(8192),Y(8192),Cos(2048)
120 REDIM X(0:Mf-1),Y(0:Mf-1),Cos(0:Mf/4)
130 MAT X=(0.)
140 MAT Y=(0.)
150 T=2.*PI/Mf
160 FOR Ms=0 TO Mf/4
170   Cos(Ms)=COS(T*Ms)    ! QUARTER-COSINE TABLE
180 NEXT Ms
190 Ta=Gp*Muf*Muf
200 IF A=0. THEN 220
210 Tb=Muf*Muf/(A*Bs*Bs)
220 T=FHEXP(2.*A-Ta)
230 Tc=T*Ta
240 Td=Ta*Ta
250 Delf=.1*Ta/(2.*PI)    ! INCREMENT IN FREQUENCY
260 COM A,Bs,Zc,Ta,Tb
270 X(0)=.5*(1.-T)        ! TRAPEZOIDAL RULE
280 FOR Ns=1 TO N
290   Corr=FNKO1(Ns*Delz)  ! CORRELATION  $k_0(\text{zeta}) - k_1(\text{zeta})$ 
300   IF ABS(Corr) < 1.E-30 THEN 340
310   Ms=Ns MODULO Mf      ! COLLAPSING
320   X(Ms)=X(Ms)+Corr
330 NEXT Ns
340 PRINT "Final value of Corr =";Corr;"   Ns =";Ns
350 MAT X=X*(Delz)
360 CALL Fft14(Mf,Cos(*),X(*),Y(*))

```

```

370  GINIT
380  PLOTTER IS "GRAPHICS"
390  GRAPHICS ON
400  WINDOW -4,2,-70,30
410  LINE TYPE 3
420  GRID 1,10
430  LINE TYPE 1
440  FOR Ms=1 TO 2000
450  F=Ms*DeIf          ! FREQUENCY
460  W=2.*PI*F
470  W1=Tc/(Td+W*W)      ! SHARP SPECTRAL COMPONENT
480  T=Mf*DeIz*F
490  Ns=INT(T)
500  Fr=T-Ns
510  W2=Fr*X(Ns+1)+(1.-Fr)*X(Ns) ! BROAD SPECTRAL COMPONENT
520  PLOT LGT(F),10.*LGT(W1+W2)
530  NEXT Ms
540  Ns=MAX(Ns,1)
550  FOR Ms=Ns TO Mf/2
560  F=Ms/(Mf*DeIz)      ! FREQUENCY
570  W=2.*PI*F
580  W1=Tc/(Td+W*W)
590  W2=X(Ms)
600  T=W1+W2
610  IF T>0. THEN 640
620  PENUP
630  GOTO 650
640  PLOT LGT(F),10.*LGT(T)
650  NEXT Ms
660  PENUP
670  PAUSE
680  END
690  !
700  DEF FNExp(Xminus)    ! EXP(-X) WITHOUT UNDERFLOW
710  IF Xminus>708.3 THEN RETURN 0.
720  RETURN EXP(-Xminus)
730  FNEED
740  !
750  DEF FNVol(Zeta)      ! CORRELATION ko(zeta)-k1(zeta)
760  COM A,Bs,Zc,Ta,Tb
770  E1=FNExp(Zeta)+Zeta-1.
780  T=Bs*Zeta
790  E2=FNExp(T)+T-1.
800  Rho=MAX(0.,1-Zeta/Zc) ! TRIANGULAR RHO
810  T=Ta+E1+A*(2.-Rho)-A*Rho*FNExp(Tb*E2)
820  RETURN FNExp(T)-FNExp(Ta*(Zeta-1.))+2.*A)
830  FNEED
840  !
850  SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; A SUBS
860  ! SEE APPENDIX A.3

```

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